SERVOMECHANISM FUNDAMENTALS

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by Henri Lauer, Robert Lesnick

and Leslie E. Matson

ALL OF THE ENGINEERING

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SERVOMECHANISM FUNDAMENTALS

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PREFACE

The purpose of this book is to introduce the reader to the principles underlying the theory of servomechanisms. It has been written primarily for the benefit of the engineering student and the practicing engineer, and therefore the approach to the problems of servomechanisms has been made with an eye toward their needs.

For the purpose of presenting the principles of servomechanisms in an orderly fashion, a rather detailed derivation has been made of the basic properties of servo control devices and their relation to the physical principles that govern their operation. In order to make this book useful to practicing engineers, a number of examples, exercises, and problems similar to those encountered in the field are given, together with the working formulas, curves, and diagrams necessary for their solution. In this manner, it is hoped that the reader will get a clear idea of the order of magnitude of the various elements involved. The system of units used has been chosen with a view toward direct application to the field problems. A few test procedures that might be helpful in checking the design and performance of a servo system have been outlined.

In this book, particular emphasis is placed on the transient analysis of elementary servomechanisms, because it is the authors' belief that this approach not only is highly useful in the appraisal of servomechanisms encountered in engineering practice, but also gives a thorough grasp of the problems that arise in their design. Classical methods are used in solving the differential equations encountered, rather than the more advanced processes of operational calculus, as it is believed that these methods more readily afford an understanding of the physical phenomena involved. For those whose memories have grown dim regarding these methods, the solutions are carried out in considerable detail.

An introduction is made, in this book, to the transfer-function or frequency analysis of servomechanisms. The authors have tried to bring out the relation between this type of analysis and transient analysis. A complete study of transfer-function analysis is, however, beyond the scope of this introductory book.

In short, this book is an attempt to discuss the operating features of servomechanisms and set forth systematic procedures for the design of such devices. These procedures consist essentially in determining the characteristics of component elements of the mechanisms required to obtain the desired performance. They do not, however, cover the design methods of such component elements as amplifiers, motors, gears, etc., the combination of which makes up a servomechanism. These are fully

described in specialized, related literature. It is the coordination of these elements into a complete system that constitutes the design of a servomechanism.

Owing to wartime security conditions, the facts discussed in this book were studied independently in many laboratories, often along parallel lines, leading to similar conclusions. Only a few of these investigations were published when this book was being written, and the attempt of the authors to mention the workers prominent in establishing the fundamental theory in this field is limited to those known to them at the time. The authors regret any credit omission that may have occurred under these circumstances.

The authors' experience in developing and designing servomechanisms, as well as in teaching and writing on this subject, during the war, under the auspices of the Radio Corporation of America, has formed the basis of the presentation of this book. They wish to thank those at RCA who have encouraged them in writing this book.

It has been the authors' aim not only to set forth the principles of the art of servo control, but to provide the engineer with a systematic design procedure for fulfilling specified operating requirements. If this aim has been met, it is the authors' hope that they have thereby contributed in some small measure in furthering the advance of this ever more important branch of engineering.

THE AUTHORS.

CAMDEN, N. J.,
January, 1946.

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SERVOMECHANISM FUNDAMENTALS

CHAPTER I

ELEMENTARY FORMS OF CONTROL SYSTEMS

. . . Aladdin had no sooner rubbed the lamp, than in an instant a genie of gigantic size appeared before him, and said in a voice like thunder, "What wouldst thou have? I am ready to obey thee as thy slave" . . . [1]

Out of the depths of history appears one of the most potent features of civilization, man's ability to utilize the forces of nature for performing physical tasks far beyond the capabilities of his own strength. It is the process of applying these forces, releasing, stopping, and properly governing their action, which is designated under the name of *control.*

The concept of control involves more than the transformation of forces by means of levers or other devices, or of energy from one form into another. It implies the conditional use of an external source of power, in most cases substantially greater than that employed for the control operation. Thus, after a load has been attached to a horse, the mere word or gesture of a man will cause the load to be carried from one location to another. The energy expended in displacing the load is far greater than that expended in uttering the command, *i.e.*, in governing or controlling the force that actually performs the work.

Almost unlimited quantities of mechanical or electrical energy can be controlled by opening or closing a simple valve or switch. Through such devices, suitably arranged, only a trifling amount of energy is required to control ships, locomotives, power plants, mills, presses, and machines of all kinds, which operate at considerably higher power levels than the controlling agent. A few typical examples of such systems are given at the end of this chapter.

Generally speaking, control systems are devised to regulate or govern a flow of energy. They vary in arrangement and complexity, depending on the nature of the functions they are intended to perform and on the required speed and accuracy of their operation. They differ, also, according to whether their operation is independent of or dependent on the flow of controlled energy. In order to facilitate an understanding of the principles involved in these various kinds of control systems, a few

[1] Adapted from "The Arabian Nights," Rupert S. Holland, ed., Macrae-Smith Company, Philadelphia.

very simple types are described in the following paragraphs. Starting with elementary devices, essential components of control systems are discussed to illustrate their mutual operating relations. This introductory discussion constitutes a general sketch of the subject, which will be treated in greater detail in the chapters that follow.

Elementary Control Devices.—One of the simplest control systems is illustrated in Fig. 1.1. It consists of an electric generator or battery A

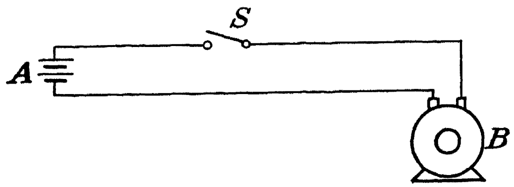

Fig. 1.1.—Discontinuous electrical control system.

connected through a switch S to a load B which may be a motor, an electric light, a heater, or the like.

Closing the switch completes the circuit and allows energy to flow from the battery to the load, in which it performs useful work. Opening the switch stops the motion of electricity in the system. Thus, operation of the switch controls the flow of energy from the generator to the load, making it either zero or a value that is determined by the characteristics of the circuit.

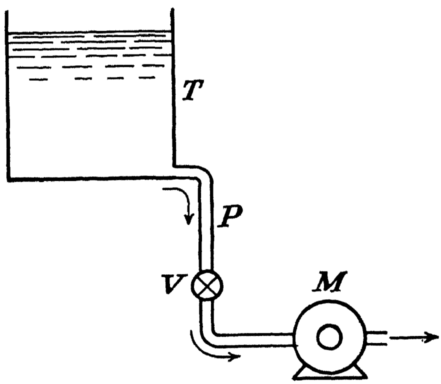

Fig. 1.2.—Continuous hydraulic control system.

Crude as it is, a control of this type embodies, nevertheless, some of the features of more complicated systems. In particular, the energy required for closing or opening the switch bears no relation to the amount of power thereby released or controlled in the circuit; the two are entirely independent of each other. Also, in the present instance, the controlling energy is mechanical, while the controlled energy in the circuit is electrical; and the energy ultimately obtained from the output load may be mechanical, luminous, or calorific, depending on the nature of the load.

In other words, the controlling and controlled energies may be of different forms.

A second system, substantially equivalent to the first, from the operating standpoint, is shown in Fig. 1.2. A large water-filled reservoir T is provided with an outlet near its base. This outlet is connected through a pipe P to a hydraulic motor M located at a lower level than the tank. A valve V is inserted at some point along the pipe P. Opening the valve allows water to flow from the tank down the pipe and through the motor, causing the latter to rotate and perform useful work. The potential energy of the water in the tank is transformed into mechanical energy in the motor, and the energy flow from the tank to the motor is started and stopped, *i.e.*, controlled, by the operation of the valve.

As in the first example, the work required for operating the valve is independent of the rate of energy flow between the tank and motor and may be of an entirely different order of magnitude.

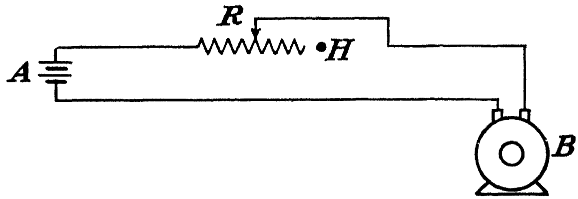

FIG. 1.3.—Continuous electrical control system.

Contrary to the case illustrated in Fig. 1.1, the control operation here need not be limited to turning the energy flow on or off. In the system shown in Fig. 1.2, there is an infinite number of possible positions, between the fully open and closed positions, to which the valve V may be adjusted. This allows the rate of energy flow to be adjusted to any desired value between the limits afforded by the system. The device thus constitutes a form of gradual or *continuous control.*

The electromechanical system of Fig. 1.1 may be modified to constitute a form of continuous control. This is shown in Fig. 1.3. In place of the switch S of Fig. 1.1, a rheostat R is inserted between the generator and the load. When the sliding contact of the rheostat is in the position H, the circuit is open, and no energy flows in the system. Moving the contact to the left closes the circuit. However, at first, the full resistance of the rheostat is in series with the circuit. Further displacement of the contact toward the left gradually reduces the circuit resistance and correspondingly increases the current and rate of energy flow. Here again, the mechanical energy involved in displacing the slider, which is the controlling energy, is entirely independent of the controlled electrical energy operating in the system and its load.

A type of rheostat more readily responsive to rapid variations of the

controlling energy may, for instance, be represented by a carbon microphone, as shown at C, Fig. 1.4. When the microphone is connected in series with a battery A and a telephone receiver or loud-speaker B, variations of pressure on the microphone diaphragm control the current flowing in the circuit. The energy producing the vibration of the diaphragm is independent of that which operates in the loud-speaker circuit. With a suitably large battery voltage, the energy flowing in the circuit may be considerably greater than that which operates the microphone. The sound vibrations produced by the loud-speaker are then amplified reproductions of the controlling vibrations impressed on the microphone. Similar amplifying features, in one form or another, are utilized in the majority of control systems, where the controlling and controlled elements of the system operate at entirely different energy levels.

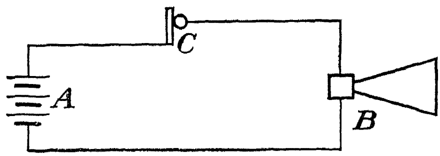

FIG. 1.4.—Electroacoustical control system.

Open- and Closed-cycle Control Systems.—It should be noted at this point that the operation of the control device, such as the switch in Fig. 1.1, *may* or *may not* be independent of the result that such operation produces at the load end of the system. In other words, the control may be operated independently from, or in accordance with, some condition at the load end of the system, whereby a demand arises for the particular kind of energy developed in the load element.

When the control operation is independent of the result obtained, the system is called an *open-cycle system.* When the control operation is a function of the result, the arrangement constitutes a *closed-cycle system.* The operating sequence then implies a measurement of the result. This measurement is followed by a corresponding action of the control. Both open- and closed-cycle systems may be operated either manually or automatically.

As an example of an open-cycle control system, a toaster may be operated by an ordinary clock-driven timing mechanism which controls the time intervals during which the heating element is energized. This control action, however, is entirely independent of the temperature and condition of the piece of toast. Consequently, line-voltage variations and previous operation of the toaster will result in different amounts of toasting for the same time cycle setting of the machine.

Another open-cycle control system is illustrated in Fig. 1.5. An oven B is heated electrically by a heating element H, which is connected through a switch S to an electric battery A. The oven temperature is

indicated on a thermometer T. In order to maintain a constant temperature within the oven, the switch would have to be closed whenever the thermometer indication fell below the assigned value and opened when the indicated temperature exceeded this value. The switch may be turned on and off by an operator observing the thermometer and acting in accordance with its indications. The *result* of closing and opening the control switch, as shown by the measured temperature of the oven, then governs

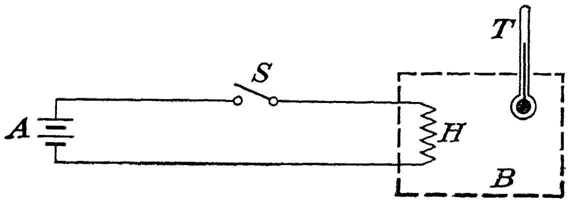

FIG. 1.5.—Temperature-control system.

the operation of the control switch. And if the operator is considered as a part of the system, this becomes a closed-cycle control system.

An equivalent, automatic, closed-cycle temperature regulating system is shown in Fig. 1.6. Instead of a thermometer, the temperature-sensitive device T is a coiled spiral of thermostatic metal strip (*e.g.*, a bimetal strip). One end of the spiral is secured in a fixed position. The other end opens or closes a contact S whenever the strip expands or contracts as the temperature rises above or falls below a predetermined value. When the contact is closed, current is supplied to the heater by the battery A. The oven temperature then increases gradually, causing the thermostatic strip to expand and open the contact S. This cuts off the electric current, and the oven cools down. The thermostatic strip then contracts and closes the contact S, causing the current to flow and the temperature to rise again.

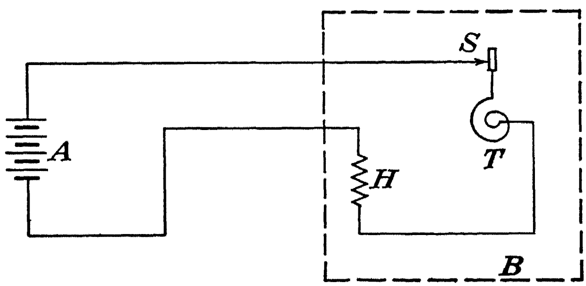

FIG. 1.6.—Automatic temperature-control system.

Similarly, in the toaster previously described the time intervals during which the heating element is energized may be controlled to be a function of the end result. As known, the amount of heat supplied to

the toast is a function of time and temperature. Thus by making the toasting time an inverse function of the operating temperature, as measured by a bimetal strip, the operation is made dependent on the temperature of the machine, thus producing even toasting.

Position-control Systems, and Use of a Differential Device.—The fundamental principles embodied in regulating systems like those described in the preceding paragraphs may be applied also in other types of control systems and in particular in systems for controlling the position of some object.

To illustrate this application, consider first the simple system shown in Fig. 1.7. An electric motor B is connected to a battery A through a double-throw switch S. The shaft of the motor is mechanically linked through a rack and suitable gears to an object or load J. When the switch is open, the motor is stationary. When the switch is closed in position a, the motor turns, say clockwise, and drives the load toward the left. With the switch closed in position b, the motor turns in the opposite

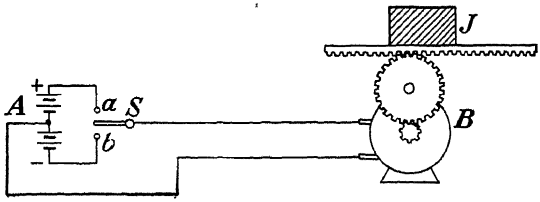

FIG. 1.7.—Discontinuous position-control system.

direction, driving the load toward the right. The device shown in Fig. 1.7 is an open-cycle, discontinuous-control system, similar to the device shown in Fig. 1.1.

The distance over which the load J is displaced may be measured directly at the load. However, if the load is located at some remote or inaccessible place, it may not be convenient or possible to make such a direct measurement. A light cable or cord may then be attached to the load, through which the displacement of the load is transmitted to a measuring or indicating instrument.

This remote position indicating instrument may be modified to predetermine and to control the displacement of the load. This is shown in Fig. 1.8, in which the load J, battery A, switch S, motor B, and associated gears are arranged in exactly the same manner as in Fig. 1.7. However, in addition to these elements, a cord W is attached to the load J at P and passed over two stationary pulleys C and E and a movable idler pulley D. This idler pulley is mechanically connected to the switch arm S, as shown by the dotted line in the diagram.

Before describing the operation of such a system, it will be noted that, if the free end M of the cord is kept stationary any displacement of the

load toward the right causes the movable pulley D to be displaced upward, while a displacement of the load toward the left produces a downward displacement of this same pulley. Similarly, if the load is fixed, a displacement of the free end M of the cord toward the left or the right causes a corresponding displacement upward or downward, respectively, of the pulley D. Thus, a displacement of *either* end M or P of the cord *toward* the pulley D causes the latter to move downward, while a displacement of *either* end of the cord *away* from the pulley D causes the pulley to move upward. The displacement of the movable pulley is equal to one-half the displacement of the particular end of the cord moved.

It follows that if *both* ends M and P of the cord are displaced by equal amounts, and in the same direction, the position of the pulley remains unchanged. But, if the displacements of M and P are unequal, the pulley will be displaced from its original position by an amount equal to

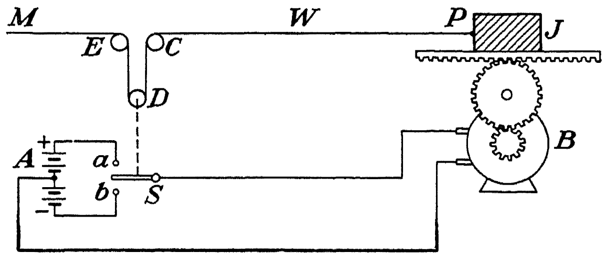

FIG. 1.8.—Automatic, discontinuous position-control servo system. (Note: E and C are fixed-pivot pulleys, and D is a movable differential idler pulley.)

one-half the algebraic difference between the displacements of M and P. In other words, if the distance between the ends M and P is the same before and after the displacements of these two ends, the position of the pulley D remains the same. But if the displacements of M and P are such that the distance between them is greater or smaller than it was originally, the pulley will be displaced upward or downward, respectively, from its original position by an amount equal to one-half the difference between the original and the final distance.

The pulley combination thus constitutes a *differential device* by which the *relative positions* of the two ends M and P of the cord are compared. The *deviation* of the pulley from its original position is an indication of the magnitude and direction of any variation in the relative positions of the ends.

In order to describe the operation of the system illustrated in Fig. 1.8, suppose that the free end M of the cord is suddenly moved toward the right by 2 in. Since the end P of the cord is stationary, the pulley D moves 1 in. downward. This closes the contact b of the switch S, and

the motor B starts rotating in a direction that, as stated in regard to Fig. 1.7, drives the load J toward the right. The end P of the cord is thereby pulled to the right, causing the pulley to move upward. When the pulley has thus been raised 1 in. (which corresponds to a 2-in. displacement of the load toward the right), the switch S opens, and the motor stops.

Thus, a motion of the end M of the cord to the right results in an equal motion of the load J to the right, and after this result is obtained the system automatically ceases to operate. During this process, no energy is transferred through the cord from the driving or input end M to the output end P and load J.

Similarly, if the input end M of the cord is moved 2 in. to the left, this motion will be taken up entirely by a 1-in. upward displacement of the pulley D, without any energy being transferred through the cord to the output end P and load. However, the displacement of the pulley closes the contact a of the switch S, and this causes the motor to rotate in such a direction as to drive the load to the left. This, in turn, allows the pulley to move downward, and when the load has moved 2 in. the pulley is returned to its original position, opening the switch and stopping the operation of the system.

In effect, the system always operates in such manner as to restore the switch to its original open position.

The same reasoning holds true for larger or smaller displacements of the input end M of the cord. Thus, a 1-in. movement of M to the right causes a 1-in. movement of the load J to the right. Successive displacements of M in the same direction cause equal successive displacements of the load in the same direction.

It follows that a correspondence is established by the system described between the positions of the input end M, on the one hand, and the output end P and load on the other. The position of M constitutes a standard, to which the position of P and connected output load is made to correspond through the operation of the system. This correspondence between the positions of M and P may also be described by saying that the system operates in such a way as to maintain or restore to a constant value the distance between the ends M and P of the cord.

The system tends to stabilize the load J in the position assigned to it at the input end M. Suppose, for example, that the load J is subjected to a strong wind, which tends to displace it toward the left. If the end M of the cord is kept stationary, a movement of J and P toward the left displaces the pulley downward. This, in turn, closes the switch contact b and starts the motor rotating in such a direction as to drive the load toward the right. The pulley D then rises and opens the switch when the load is restored to its original position.

The preceding discussion was predicated on the assumption that one

of the contacts *a* or *b* of the switch will close, irrespective of how small the displacement of the input member *M* may be. Actually, the spacing between the arm and stationary contact points of the switch cannot be made infinitely small, and therefore a finite displacement, small as it may be, is required to operate the switch. This introduces some slight backlash or *play* in the system. In other words, the accuracy of the correspondence established by the system between the stable positions of the input and output members is limited to the minimum displacement required to open or close the contacts of the control switch. This limitation does not, however, alter the principles underlying the operation of the system.

Discontinuous-control System.—In the control system shown in Fig. 1.8, it was found that a finite displacement of the input member *M* results in an equal displacement (within the limitation mentioned in the last paragraph) of the output member *P* and load *J*, in magnitude as well as in direction. However, in the above description, only the initial and final positions of the input and output members were considered, without regard to the time element involved in the displacement process.

In this connection it should be noted that the system shown in Fig. 1.8 is a *discontinuous* type of control; that is to say, as soon as the input member is displaced from its original position and the switch is closed the full voltage of the corresponding section of the battery is applied to the motor. The motor then develops its full torque and accelerates the output member and load at a rate that is determined solely by the electrical and mechanical characteristics of the motor and load, independent of the speed of the input member *M*.

If the speed of the input member is greater than that of the load, the motion of the load will not restore the differential pulley *D* to its original neutral position until some time after the input member *M* has stopped in its final position. On the other hand, if the speed of the input member is smaller than that of the load, the motion of the load will, in spite of the motion of the input member, cause the differential pulley to return to its original position and open the switch. The motor then slows down until the continued motion of the input member again displaces the pulley and closes the switch, when the full motor torque is once more applied to the load. Under these conditions of low input speed and high output speed, a constant-speed input motion results in a periodically variable output speed motion. It follows that, while the respective ultimate displacements of the input and output members are equal, the instantaneous input and output speeds are different. Slow speed and accuracy are thus not readily obtainable with discontinuous control.

Continuous-control Systems.—The control system of Fig. 1.8 was shown to establish a correspondence between the input and output dis-

placements without, however, affording accurate means of output speed control. The following slightly different system provides this additional feature.

As shown in Fig. 1.9 a potentiometer R is substituted for the switch S in the system of Fig. 1.8. The discontinuous *on-off* type of control provided by the switch is here replaced by a gradual or continuous type of control furnished by the potentiometer. The fundamental difference introduced by this new system is that the displacement of the potentiometer slider is equal to that of the differential pulley D, and the resulting voltage applied to the motor B becomes *proportional* to this displacement.

Suppose that the two sections of the battery A, Fig. 1.9, are equal. When the potentiometer slider is in its mid-position, no voltage difference appears between the slider and the center tap of the battery, and therefore no voltage is applied to the terminals of the motor. This *zero-voltage*

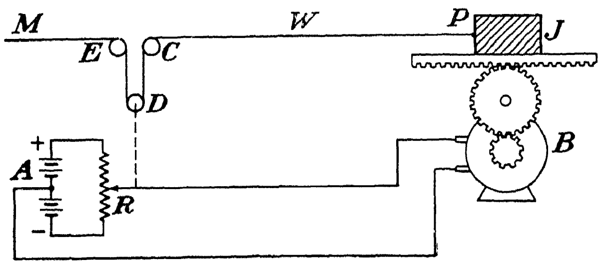

FIG. 1.9.—Automatic, continuous position-control servo system. (Note: E and C are fixed-pivot pulleys, and D is a movable differential idler pulley.)

position of the slider corresponds to a definite position of the differential idler pulley D, to which the slider is connected mechanically, and to a particular relative position of the ends M and P of the cord W.

If the pulley and potentiometer slider are in some way disturbed from this *zero* position, a voltage proportional to the displacement will be applied to the motor. This voltage is positive or negative, depending on whether the displacement is upward or downward. The motor will, accordingly, rotate in one direction or in the other. Moreover, the characteristics of the motor may be made such that its torque is proportional to the applied voltage. Therefore, the motor torque will be proportional to the displacement of the pulley.

Suppose that the above system is first at rest, with the differential pulley and potentiometer slider in zero, or centered, position. Then, let the input member M be suddenly set in motion at constant speed toward the right. The pulley and the potentiometer slider connected to it will then move downward, applying a voltage to the motor. The motor starts turning and drives the load toward the right. If the speed

of motion of the load is less than that of the input end M of the cord, the pulley and potentiometer slider continue to move downward as M is being moved to the right. This applies a greater voltage to the motor, so that the motor torque and consequently the motor speed increase accordingly. If then the displacement speed of the load comes to exceed that of the input member M, the pulley D will be driven upward, reducing the applied voltage and slowing down the motor. A position of equilibrium will thus be reached, for which the input and output speeds (that is to say, the speeds at M and at J or P) are equal. If the input speed is increased or decreased, the output speed increases or decreases by the same amount, through a corresponding change in the position of the differential pulley.

The torque developed by the motor and the resulting driving force applied to the load serve to offset the retarding force produced by such friction as may be present in the moving parts of the system. In the theoretical case in which no friction exists in the system, no torque would be required from the motor after the load has been brought up to a speed equal to the constant speed at which the input member M is being driven.[1] The pulley and potentiometer then automatically return to the zero position.

Error.—It was shown that the system of Fig. 1.8 establishes a position correspondence between the input and output members, whereby the distance between the ends M and P is the same *before* and *after* a limited displacement of the input member M to the right or left.[2] It may readily be understood that this holds true also in the case of Fig. 1.9.

However, *during* the motion of the system from its original to final positions, a voltage must be applied to the motor in order to accelerate the load up to the input speed and then sustain it at that speed against such deceleration as may be produced by the output friction of the system. If friction, or its equivalent, should be absent from the system, torque is required from the motor during the accelerating period only. If some form of friction or external load is present, constant-speed motion of the input and output members requires a constant voltage to be applied to the motor during the entire period of motion to overcome either of these retarding torques.

[1] Recalling Newton's law, a force is defined as the *accelerating* or *decelerating agent of motion*. After a mass (represented here by the load, gears, and motor rotor) has been set in motion, it keeps moving at constant speed in absence of any applied force.

[2] The accuracy of this correspondence was shown to be limited by the play occasioned by the spacing between the arm and stationary contacts of the switch S. In the case of Fig. 1.9, any such discrepancy is practically eliminated if the rheostat R is uniformly and closely wound with fine resistance wire. It may, indeed, be eliminated completely if a slide-wire rheostat is used. Only such limitation then arises as may be due to backlash in the gears or to the presence of static friction in the system.

As described before, such an applied voltage results from a shift of the differential pulley from its *zero* position to a new position. In other words, the pulley automatically assumes the position necessary to supply the voltage required by the motor to deliver the load torque.

In a system arranged in the manner shown in Fig. 1.9, a motion of the input member M and consequently of the load J toward the left causes a finite upward displacement of the pulley D, while a motion toward the right causes a similar downward motion of the pulley. In either case, the deviation of the pulley from the zero position produces a change in the length of the portion of the cord between the pulleys C, D, and E. The distance between the ends M and P during the motion is therefore different from that corresponding to the zero position of the pulley, when the system is at rest. Thus, while the distance between the ends M and P has the same value *before* and *after* a displacement of the input member M, it differs from this value *during* the motion of the system, as long as the motor is made to develop torque and apply an accelerating force to the output load. An *error* is thereby introduced in the relative positions of the input and output members of the system, which disappears when the system is again brought to rest.

The displacement of the differential pulley as well as the voltage at the potentiometer slider is directly proportional to this relative input-output position error. This voltage is therefore a measure of the error, and is often designated as *error voltage.*

Damping, Oscillation, and Speed of Response.—In many practical applications it is important to reduce the error, described in the preceding paragraph, to as small an amount as possible. Methods for achieving this are discussed in detail in later chapters, but an idea of the problem may be given here in order to outline some of the phenomena involved.

It follows from the above discussion that the error which exists between the relative positions of the input and output members during the steady-state constant-speed motion of the system is due to the presence of load or friction. The reason was found to be that such loads develop an output retarding force which must be offset by a driving force obtained from the motor. This, in turn, requires that voltage be applied to the motor, which is obtained through a displacement of the differential pulley from its zero position. It is this displacement that is produced at the cost of the error.

It thus appears that the error will be smaller, as the output friction of the system is smaller. While a small output friction causes only a small steady-state error as long as the input speed is constant, it allows the system to oscillate whenever the input speed is changed: the less the friction, the greater and more persistent will be the oscillations when the input member is suddenly started and stopped.

To illustrate this point, assume that the system is at rest and that the differential pulley and potentiometer are in the zero position. Then let the input member suddenly be given a small limited displacement by moving it rapidly toward the right and then stopping it. If the motor and output load are stationary, the differential pulley moves downward by an amount corresponding to the input displacement. This applies a voltage to the motor, and the motor then drives the load toward the right, causing the differential pulley to move upward toward the zero position. The applied motor voltage thereby tends toward zero.

If the output friction is small, the original acceleration of the motor may store sufficient kinetic energy in the inertia of the moving parts of the system to cause the output member to *overshoot* the position assigned to it by the new input member position. The differential pulley is, in this case, driven upward through and beyond its zero position, applying a reversed voltage to the motor. The motor torque then reverses and accelerates the load in opposite direction, and a succession of reversals of the motor torque and output load displacement may occur.

The resulting oscillation, also called *hunting*, may last for some considerable time after the initial impulse displacement of the input member, much like the vibration of a gong after the impact or shock of the hammer producing it. During such oscillation, the position of the output member of the system is periodically out of correspondence with that of the input member, impairing the accuracy of the control.

The amplitude of hunting or oscillation decreases gradually, as the energy stored in the output inertia is spent in the friction of the system. The rate of decay of the oscillation amplitude increases with the magnitude of the output friction. This friction may be made so great that the output member reaches its final position without any oscillation, though with considerable time delay.

Also, while friction will damp out oscillations caused by varying conditions of input speed and thereby will stabilize the system, it also introduces a steady-state velocity error, whereby the position of the output member lags behind that of the input member. In the simple system of Fig. 1.9 a compromise must therefore be found for the value of friction, between the amount of permissible steady-state error and transient oscillation. Other methods of designing the system and adjusting its performance will be described in later chapters after the more elementary conditions discussed here have been analyzed more accurately.

Rotational-control Systems.—In the preceding examples of position-control systems, both the input and output members were subject to translatory motion. Systems employing equivalent or similar component elements may be used for the control of the angular position of an input and output shaft.

The diagram of Fig. 1.10 schematically represents such a system, which will be studied at greater length in following chapters. The purpose of the system is to make the instantaneous angular position of a load J follow the changing angular position of a control input shaft M. To accomplish this, the input shaft is connected to one side of a differential device D, the other side of which is connected to the load. Only a negligible amount of power is required to drive either side of this device, through which no energy is transferred from the input shaft to the load. The idler or differential member of the device assumes a position that is a function of the difference, or error, between the angular positions of the input shaft M and load J. The idler member of the differential D actuates a controller B. This controller embodies some type of drive motor and such amplifying means as may be required. It drives the load in accordance with the error indications of the differential device. The damper F shown in the diagram represents the output friction of the system.

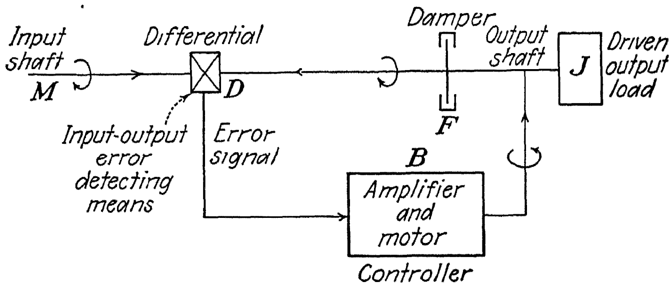

FIG. 1.10.—Block diagram of simple position-control servo system.

If the system is at rest and the input shaft is suddenly rotated, the angular difference or error between the positions of the input and output shaft will deflect the differential device from its original position. This applies a signal to the controller, which then drives the load in such manner as to reduce, or tend to reduce, to zero the error indication of the differential device. In this way, the rotation of the controller motor and its connected output load *follows* the rotation of the input shaft. Hence the name *servo* or *slave* applied to the system.

Essential Components of a Servo Control System.—It is now possible to summarize the discussion of this chapter by listing the essential components embodied in a servo control system. These components and their mutual relationships will be studied quantitatively and in detail in the chapters that follow.

The purpose of a position servo control system is to cause an output load, which may be remotely located, to follow rapidly and accurately the position of a control input shaft or member. Such a servo system is comprised of the following components:

1. An input member or shaft.
2. An output member or load.
3. A differential error detecting device, which compares the instantaneous positions of the input and output members of the system.
4. A controller embodying a motor with such amplifiers and power source as may be required and such gears and linkages as are needed to connect it to the output member and load.
5. Damping or stabilizing devices.

The functions of these various components may be defined as follows:

1. The input member instantaneous position constitutes at all times a standard with which the output member position is to be made to correspond, through the operation of the servo system.
2. The output member is that part of the system which is driven into position correspondence with the input member.
3. The differential device produces a signal proportional to any difference, or error, between the input and output positions. This signal may be a mechanical displacement, an electrical voltage, etc.

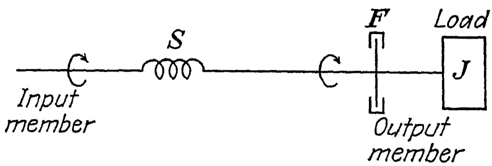

FIG. 1.11.—Direct-drive positioning system.

4. The controller is actuated by the error signal produced by the differential device. It develops and delivers to the output member and load a driving force of such direction and magnitude as to displace the load in a manner that will reduce (or *zero*) the error signal produced by the differential device.
5. The damping and stabilizing devices serve the purpose of reducing oscillations, thereby improving the performance of the system.

From a functional standpoint, it is instructive to compare a servo system, as just described, with a system, such as that shown in Fig. 1.11, in which the input member is connected directly to the output member and load through a spring S. The output inertia J and friction F are assumed to be the same in both cases, and the strength of the spring S can be so chosen that the input-output position correspondence is the same in the two systems no matter how the input member is moved. But the direct-drive system shown in Fig. 1.11 embodies no control feature and requires the full load-driving torque to be applied to the input member, from which it is conveyed directly to the load. In the servo system shown in Fig. 1.10, no torque is transmitted from the input member to the load. Instead the servomechanism compares the input

and output positions, and translates any discrepancy between these into an error signal. This signal, in turn, releases a corresponding amount of power from the energy source associated with the controller. It is this source that, energizing the controller, causes the latter to develop the necessary torque to drive the output member and load into position correspondence with the input member, by *zeroing* the differential error signal. Thus, the input member driving agent is entirely relieved of the work of driving the load, this work being performed by the controller at the expense of its associated power source.

Examples of Practical Servomechanisms.—The preceding qualitative discussion was intended to give a general outline of the arrangement and properties of simple servomechanisms. A few examples of practical applications of these devices are described briefly below.

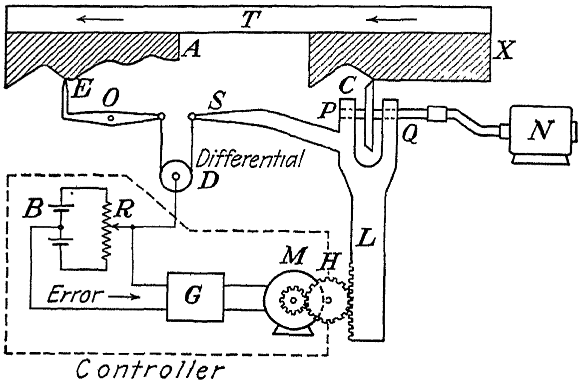

FIG. 1.12.—Automatic profile milling machine.

Profile Milling Machine.—The object of such a machine is to duplicate the shape, or profile, of a model, either for the simple purpose of reproduction, or for making a copy in a different—generally harder—material than the model. The system is illustrated in Fig. 1.12, which is sketched in very schematic form, in order to emphasize its correlation with the general pattern described before.

The model A and the piece X which is to be milled to the profile of the model are held rigidly in a common mounting support T, which may be driven from right to left either by hand or by a motor (not shown in the figure). A light feeler gauge E is applied against the profile of the model with just enough pressure to allow it to move up and down when following the shape variations of the model, as this is displaced laterally past the feeler point.

The feeler gauge is pivoted at O, and its motion is transmitted to one side of a differential device, which is shown here as a pulley and cord D associated with a potentiometer R and battery B in the same manner as

described before. The other side of the differential is connected to an extension S of the frame L, which supports two bearings, P and Q. In these bearings rotates a milling cutter C, driven by a motor N through a flexible shaft. The frame L may be displaced vertically up and down, by means of a servo motor M to which it is connected mechanically through a suitable gear and rack H. The motor M is actuated, through an amplifier G, by the error voltage developed by the differential-operated potentiometer R.

By comparison with the servomechanisms described before, it is readily apparent that the feeler gauge E represents the input member, while the frame L and cutter C constitute the output load of the servo. A downward motion of the gauge point causes an upward motion of the differential pulley resulting in a corresponding potentiometer voltage being applied to the servo motor through the servo amplifier. For a proper electrical connection polarity of the motor, the latter then drives

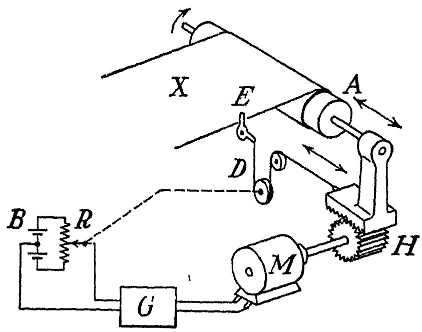

FIG. 1 13.—Automatic winding-control servo system.

the frame and milling cutter downward by an amount equal to the displacement of the feeler gauge in order to *zero* the differential voltage. Similarly, an upward displacement of the feeler point results in an equal upward displacement of the cutter.

Thus, the up-and-down motion of the cutter reproduces the up-and-down motion of the feeler, and the piece X will be milled to the same profile as that of the model A. The power for rotating the cutter is supplied by the motor N, which is independent of the servo system while that required for determining the shape or profile of the cut is supplied by the servo motor M.

Control of Winding Operation.—A similar mechanism may be adapted for controlling the edge position of strip or sheet material to be wound on drums. This is shown in Fig. 1.13, where the sheet X is being wound on a drum A. A feeler or finger E is applied lightly against the edge of the sheet, and its lateral motion, in following any deviation of the edge, operates one side of the differential gear D. The potentiometer R and

battery B associated with the differential produce the error voltage that is fed to the motor M through the amplifier G. The motor, in turn, displaces the drum along its axis, through the rack and gear H, to position the drum so as to keep the edge of the sheet in proper alignment, by zeroing the indication of the differential device.

Color-printing Register Control.—Multicolor printing requires the same sheet of paper to be placed in proper position successively under a number of plates, each plate printing to a different color. In order to ensure correct positioning of the different color impressions, a mark printed on the paper must be brought in the same position with respect to each of the several plates. Instead of a feeler gauge, a photoelectric cell is here used to produce the electrical input signal to the error detecting device. Differential indication is obtained by opposing a fixed voltage to the photoelectric cell voltage, the difference or *error* being used to

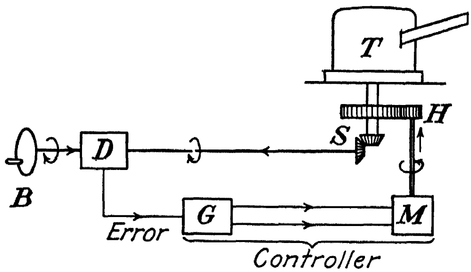

FIG. 1.14.—Gun-turret-positioning servo system.

actuate a servo motor by way of a suitable amplifier. The motor then operates in such manner as to *zero* the error by properly positioning the sheet bearing the register mark.

Similar devices are used to control the tension of cloth, paper, or other materials in continuous strip processes.

Military Applications.—Servomechanisms are used extensively in military applications, such as the speedy and accurate positioning and training of guns, turrets, radar and radio antennas, steering of ships, automatic pilots, etc. A simple gun-turret training system is illustrated in Fig. 1.14. The turret T is connected to a motor M through a gear H. A position indicator S, which may be a synchro repeater as described in the following chapter, is mechanically linked to the turret and transmits a signal to the differential device D. This differential device also receives a positioning signal from the handwheel B, which constitutes the input member of the system, and determines the position into which the turret is to be driven. The output signal of the differential, proportional to the difference or *error* between the positions of the handwheel and turret, actuates the amplifier G and thence the servo motor M. The latter then

operates and drives the turret in such a position as to *zero* the differential error signal, when both the turret and handwheel are in proper position correspondence.

Conclusion.—After defining the concepts of control and of open- and closed-cycle servo control systems, the means employed for accomplishing control operations have been briefly described in this chapter. Fundamentally, a control system includes comparatively few essential components. These are arranged according to a simple pattern, irrespective of the particular application of the servo system considered.

While the basic components and their arrangement as described here are generally encountered in all types of servos, their complexity and refinements depend on the requirements of particular installations. The following chapters will show how certain problems pertaining to speed of response, accuracy, and power output can be met by a suitable proportioning of the several factors involved, as well as by the inclusion of proper damping, stabilizing, and correcting means. Even for these seemingly more complicated mechanisms, procedures will be derived that greatly simplify the systematic design of servos suited to almost any task.

This discussion will particularly relate to mechanisms for position control. By thus centering the problem on one particular type of control, servomechanism control, a coherent treatment is possible, which readily brings out the purpose and characteristics of the various kinds of devices employed. As will be shown later, this does not impair the generality of the conclusions reached, which will be found to be applicable to other types of control, such as process control, regulating controls, and the like.

Only a few practical examples of servomechanisms were described in this chapter. They should be sufficient, however, to give an idea of the variety of duties for which these devices can be used, and which could hardly be accomplished by other means. Outstanding features of the results obtainable through servomechanisms are the facts that the controlled and controlling elements of the system may be widely separated from each other, that the work performed may be many times greater than that required for the actual control operation, and that often complicated processes can be made to take place according to predetermined standards with speed and accuracy. Thus, through their adaptability to applications of almost unlimited scope and intricacy and the tremendously increased powers of control they are bestowing, servomechanisms are proving to be the slaves that faithfully fulfill the orders of our modern Aladdins.

CHAPTER II

SERVO SYSTEM FOLLOW-UP LINKS

In the preceding chapter typical servo systems were described in a general way in order to define essential components of such systems in their mutual relationships. Certain of these elements will be discussed below at somewhat greater length under the name of follow-up links. The performance requirements of prime movers or servo motors as well as amplifiers will be discussed in later chapters in the course of a more detailed analysis of servo systems.

The devices to be described as follow-up links serve primarily as the input-output error detecting means of the servo system, as defined in Chap. I. In its simplest form, a follow-up link is essentially a *differential device,* which produces an indication of the magnitude and direction of the *error* or discrepancy between the instantaneous positions of the input and output members of the system.

The differential may be either an electromechanical or a purely mechanical device. In the latter case, it may be used in conjunction with a *translating device*, or *transducer*, that converts the error indication of the mechanical differential device into an electrical voltage. This voltage is then capable of energizing an electric servo motor either directly or through some form of amplifier.

Mechanical Differential Devices.—Mechanical differential devices are best used when the various parts of the servo control system are located in the vicinity of each other, and mechanical linkages can be established between them without undue difficulty. When the input and output members of the system operate the differential device through translatory motion, some form of pulley arrangement may be used, such as that described in the preceding chapter. Lever combinations, like a pantograph, for example, may also be employed, but they need not be described here.

When the input and output members have rotary motions, a differential pinion and gear combination is frequently used. This is illustrated below in order to show clearly the manner in which an *error* indication is obtained from such an arrangement. Referring first to Fig. 2.1, consider two bevel gears A, B of equal diameters and numbers of teeth and a pinion C meshing with these gears. The two gears A, B are locked on their respective shafts M, E, which can rotate independently from each

other in bearings G, K of a mounting frame F. The pinion C and its shaft can rotate on a bearing in the frame F.

Under these conditions, suppose that the shaft E and its gear B are held stationary, while the shaft M and gear A are rotated by some given angle in the direction of the arrow m. The motion of the gear A then causes the pinion C to rotate as shown by the arrow n, and this, in turn, drives the frame F into the position F', in the same direction (arrow p) as the shaft M (arrow m). The angle of rotation of the frame F is equal to one-half the angle of rotation of the shaft M. Thus, if the shaft M is rotated by, say, 90 deg., the frame F will turn by 45 deg.

If at this point the shaft E is turned in opposite direction (arrow r) while the shaft M is held stationary, the frame F will rotate in a direction

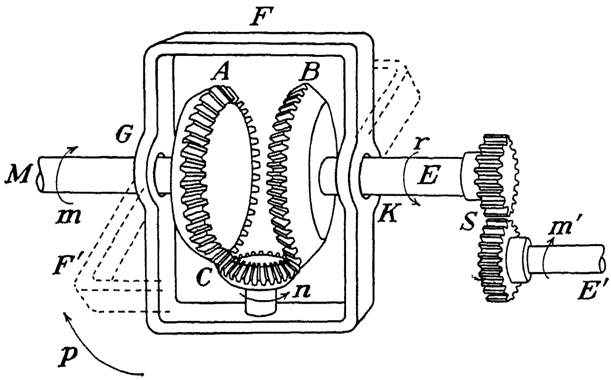

FIG. 2.1.—Differential gear train.

opposite to that shown by the arrow p, and a 90-deg. rotation of the shaft E returns the frame from the position F' to its original position F.

Thus, simultaneous and equal rotations of the shafts M and E in opposite directions cause the frame F to remain fixed in its original position.

The shaft E may be connected to another shaft E' through a pair of identical spur gears S (1:1 gear ratio) which, without changing the relative speeds of the two shafts E, E', simply cause them to rotate in opposite directions. Simultaneous rotations of the two shafts M and E' at the same speed and in the same direction (arrows m and m') will then leave the frame F fixed in its original position. In other words, if the two shafts M and E' are rotated in such manner as to remain at all times in the same original relative angular position, the frame F will remain fixed in its original position.

Suppose now that the system, being first at rest, has its shaft M started rotating at some constant speed in the direction m, while the shaft E is temporarily held stationary. As described before, the frame F

will then turn in the direction p at half the speed of the shaft M. If, at the instant the shaft M has completed its first quarter turn (and the frame F has turned by 45 deg.), the shaft E' is also set in rotation at the same speed and in the same direction (arrow m'), the frame F will henceforth remain stationary in a position 45 deg. away from its original position. This unvarying 45-deg. deflection of the frame F thus denotes that (1) the two rotating shafts M and E' maintain a constant relative angular position, and that (2) this relative position differs by 90 deg. from the original relative position of the two shafts. The deflection of

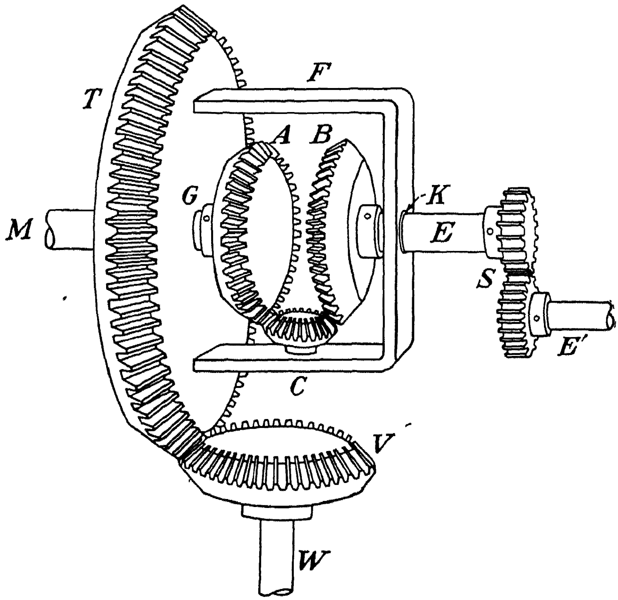

Fig. 2.2.—Differential gear with error-indicating shaft.

the frame is thus proportional to the error or discrepancy in the relative angular positions of the two shafts M and E' caused by the fact that the shaft E' did not immediately follow the motion of the shaft M. The deflection is, of course, independent of the absolute rotation speed of the two shafts.

There are many ways in which the angular position of the frame F may be transmitted to some other part of a servo system, where it will indicate the error between the input and output shafts. These may be the shafts M and E' of the differential. One method is illustrated in Fig. 2.2. This shows the same arrangement as Fig. 2.1, except that the left side of the frame F is here constituted by a bevel gear T, which meshes with a pinion V. The shaft M of the gear A rotates in a bearing located at the center of the gear T, but does *not* drive the gear T, which simply forms a part of the frame F. Any error or discrepancy in

the relative angular positions of the two shafts M and E' causes a corresponding deflection of the frame F, as before, and of the gear T, pinion V, and attached shaft W. By making the number of teeth of the pinion V equal to one-half the number of teeth of the gear T, a 2:1 step-up ratio is obtained, and the angular deflection of the differential shaft W is then equal to the angular error of the relative positions of the shafts M and E'.

It should be noted that, since the frame F is free to swing on its bearings G and K, no power is transmitted through the device from either one of the shafts M, E' to the other. Nor is any energy absorbed by the device from the energy sources driving the shafts M and E', other than the energy required to swing the frame F and differential shaft W and that consumed in the friction of the bearings and gears. The weight and friction of the gears, frame F, and whatever translating device may be connected to the shaft W must be kept to a minimum in order not to transmit any torque to the input and output shafts M and E' of the system. The differential may thus be considered as a measuring device, which simply compares the instantaneous relative positions of the shafts M and E'.

Translating Devices.—The error indication produced by such mechanical differential devices as were described in the preceding section (Fig. 2.1 and 2.2) and in Chap. I (Fig. 1.9) consists in a displacement or deflection of the differential member (frame F of Fig. 2.1 and 2.2, or pulley D of Fig. 1.9) of the device. When the system load is driven by an electric motor, this mechanical error indication must be converted into an electrical signal voltage. The magnitude and polarity of this voltage must correspond to the magnitude and direction of the error indication, and the voltage is then used to control the rotational speed and direction of the motor.

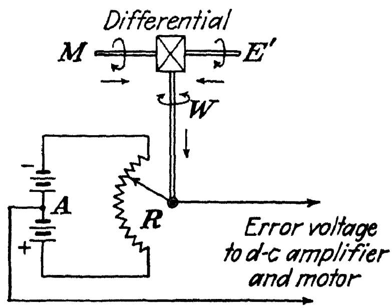

FIG. 2.3.—Direct-current potentiometer-type error-translating device.

A simple translating device, employing a battery and potentiometer, was illustrated in Chap. I in relation to a differential pulley arrangement. An equivalent device, adapted to the differential gear-and-pinion combination just described, is shown in Fig. 2.3. It consists of a potentiometer R connected across a battery A. The rotor arm of the potentiometer carries the sliding contact and is driven mechanically by the differential shaft W of the gear-and-pinion combination shown in Fig. 2.2. The output voltage of the device is obtained between the potentiometer rotor arm and the center tap of the battery A.

When the rotor arm makes contact with the mid-point of the potentiometer resistance winding, the output voltage is zero. When the rotor arm is displaced from this zero position through a deflection of the shaft W, a continuous (d-c) voltage appears at the output terminals (rotor arm and battery center tap). This voltage is proportional to the deflection angle, and its polarity depends on the direction in which the deflection takes place. In this manner, the angular error indication produced by the differential is converted into a corresponding error voltage.

Practical values for the battery voltage and potentiometer resistance might be such that the error voltage produced is of the order of 1 volt per deg. of angular error. This voltage is too small, in most cases, to drive the servo motor directly, and it is therefore applied to a suitable amplifier, which then feeds into the motor windings. A battery and

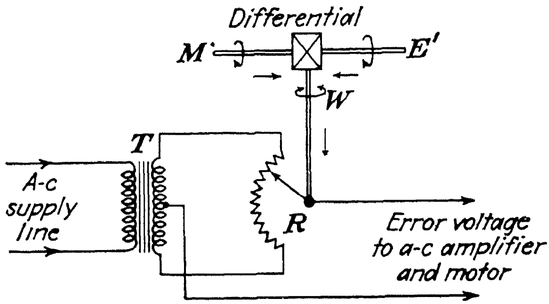

FIG. 2.4.—Alternating-current potentiometer-type error-translating device.

potentiometer capable of energizing the motor without any amplification would necessitate a potentiometer slider contact of greater current carrying capacity. In view of the greater mechanical pressure required for such a contact, excessive torque would be transmitted through the differential device to the input shaft. This is to be avoided, since the main object of a servomechanism is to drive a load without requiring that any appreciable power be delivered by the input, or control, shaft.

The continuous (d-c) error voltage obtained with the arrangement of Fig. 2.3 requires that a d-c amplifier be used. In some cases it is found preferable to use an a-c amplifier, operating from an alternating error voltage. Such an error voltage can be obtained by means of an arrangement shown in Fig. 2.4, which differs from that of Fig. 2.3 in that an a-c source is connected across the potentiometer winding in place of the battery A. Thus, referring to Fig. 2.4, the a-c supply line feeds the primary winding of a transformer T. The terminals of the secondary winding are connected to the terminals of the potentiometer resistance winding. The output terminals of the device are the rotor arm of the potentiometer and the center tap on the transformer secondary. As in the case of Fig. 2.3, there is no output error voltage when the potentiom-

eter slider is halfway between the ends of the resistance winding. However, when the slider is displaced from this zero position, an alternating voltage appears at the error-voltage terminals. The error voltage is in phase or in phase opposition with the line voltage, depending on the direction in which the slider is displaced from its zero position. This phase polarity of the error voltage indicates the direction of the angular error of the servo system, just as the d-c voltage polarity did in the arrangement shown in Fig. 2.3.

Instead of the potentiometer arrangement of Fig. 2.4, a variable coupling device may be used between the a-c line and the error-voltage circuit. This is shown in Fig. 2.5, where a stationary coil A is connected

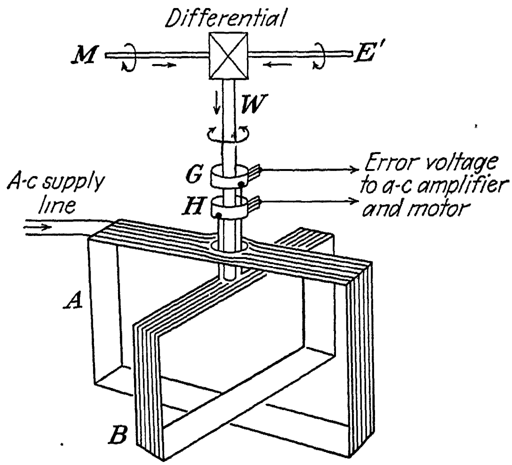

FIG. 2.5.—Variable-coupling error-translating device.

to the a-c line, while a coil B capable of being rotated about its axis is mechanically connected to the error shaft W of the differential device. Slip rings and brushes G and H connect to the movable coil B and form the error-voltage terminals of the arrangement.

When the magnetic axis of the coil B is perpendicular to that of the coil A, no voltage is induced in the coil B. This corresponds to the *zero error* position of the differential shaft W. An angular error between the shafts M and E' that operate the differential gears produces a corresponding angular deflection of the shaft W and coil B, in which an alternating error voltage is then induced by the current flowing in the coil A.

This voltage appears at the slip rings G and H and is proportional to the cosine of the angle between the magnetic axes of the coils A and B. For small error angles, such as are usually encountered in servo systems, the magnetic axes of the two coils are almost perpendicular to each other (the angle between the axes being complementary to the error angle), and the error voltage may therefore be considered, with a sufficient degree

of accuracy, as proportional to the error angle. The error voltage is in phase or in phase opposition with the a-c line voltage, depending on the direction of the angular deviation of the movable coil B from the position where its axis is perpendicular to that of the coil A.

In order to minimize any reaction between the two coils A and B, the coil B is preferably connected, through its slip rings G and H, to a high impedance load, such as the control grid circuit of a vacuum tube a-c amplifier. The current in the coil B is then negligibly small, and thus no appreciable load is applied to the shaft W. This ensures maximum performance accuracy of the differential device. The arrangement of Fig. 2.5 is not described here further, in view of its close relationship to the electrical follow-up link devices studied in the succeeding paragraphs.

It may be noted, in passing, that an arrangement equivalent to that shown in Fig. 2.5 is obtained by interchanging the a-c line and error voltage terminals. That is, the a-c line is connected to the brushes and slip rings G and H, while the error voltage appears at the terminals of the stationary coil A.

Electrical Follow-up Links.—In the systems described above the shafts M and E' through which the differential device is operated are connected mechanically to the position-controlling input member and to the output load, respectively. It is not always possible or practical to make the required mechanical connections between the several elements of the system, particularly when these elements are remotely located from each other. In order, then, that the differential device may compare the relative positions of the input and output members of the system, these positions are transmitted to the device through electromechanical position repeating apparatus, as described below. These are designated here under the name of *self-synchronous repeaters* or *synchros*, although other names are often used in practice.[1]

The external appearance of these self-synchronous repeaters resembles that of a small electric motor or generator. However, the internal construction may differ according to the intended function of the unit. The most usual types are the synchro generators, synchro motors, differential synchro motors and synchro generators, and synchro control transformers. Before describing the outstanding features of these devices, their *operating functions* in relation to a servo control system are listed below.

Operating Functions of Self-synchronous Repeaters.—1. A *synchro generator* is excited by a single-phase alternating voltage of constant peak value. It produces a set of three alternating output voltages, which are in phase or in phase opposition with the exciting voltage.

[1] Devices of this sort are designated as *Selsyns* (General Electric Company), *Teletorque* (Kollsman), and *Diehlsyn* (Diehl Manufacturing Company).

The respective phase polarities and peak values of these three output voltages uniquely represent and identify the angular position of the rotor shaft of the unit, and thus permit this to be transmitted to a remotely located part of a servo control system.

2. A *synchro motor* is energized by the three alternating output voltages of a related synchro generator, and excited by the same single-phase alternating voltage source as this generator. Under such operating conditions, the rotor shaft of the motor orients itself to the same angular position as the rotor shaft of the generator. Thus, while the synchro generator serves to convert the angular position of its shaft into a set of voltages, the synchro motor converts this set of voltages into a similar angular position of its shaft. The position of the motor shaft will then follow any variation of the generator shaft position, much as if the two shafts were interconnected mechanically.

3. A *differential synchro generator* is excited by the three alternating output voltages produced by a simple synchro generator as defined in (1) above. It produces three alternating voltages, the respective phase polarities and peak values of which identify an angular position which is the algebraic difference of the angular positions of the generator and differential generator rotor shafts. The differential synchro generator thus converts the angular positions of its own shaft and of a remotely located shaft into a resultant set of three electrical voltages.

4. A *differential synchro motor* is energized by two sets of three alternating voltages each, such as may be produced by two simple synchro generators, respectively. The angular position of its shaft is the algebraic difference of the angular positions of the shafts of the two synchro generators. The shafts of the two generators and the shaft of the motor thus perform similar functions, respectively, as the driving shafts M and E (or E') and the differential shaft W of the mechanical differential device shown in Fig. 2.2.

5. A *synchro control transformer* is excited by the three alternating output voltages of a synchro generator. It produces an alternating voltage, the phase polarity and peak value of which identify the angular difference between the positions of its shaft and that of the synchro generator. Thus, while a synchro motor as defined above in (2) converts the synchro generator output into a torque that swings the motor shaft into the same position as the generator shaft, a synchro control transformer produces no torque, but converts the generator output into a voltage identifying the direction and magnitude of the position difference between the generator and control transformer shafts.

Synchro Generator.—The function of a synchro generator as defined in paragraph (1) of the preceding section consists in producing a set of alternating voltages that identify the angular position of its rotor shaft.

When applied to a corresponding synchro motor, this set of voltages, as described later, affords the means for reproducing in the motor the angular position of the generator shaft.

The construction of a synchro generator is illustrated in the simplified disassembled view of Fig. 2.6. The machine comprises a two-pole rotor, the single winding of which connects to two slip rings R_1 and R_2. On the stator are mounted three coils with distributed windings located in slots that are evenly spaced around the stator circumference. The

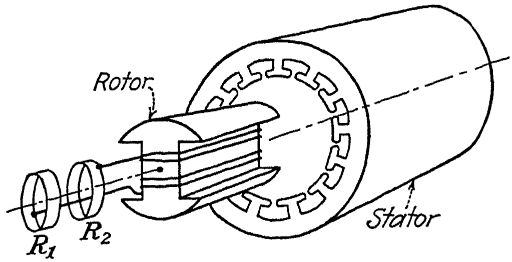

FIG. 2.6.—Synchro generator, simplified disassembled view.

magnetic axes of these stator coils are 120 deg. apart, and the coils are Y-connected and brought out to three terminals shown as S_1, S_2, and S_3 in the schematic diagram of Fig. 2.7.

When the rotor terminal slip rings R_1 and R_2 are connected to an a-c supply line, the rotor current produces an alternating magnetic field which induces alternating voltages of the same phase or in phase opposition in the three stator coils S_1, S_2, and S_3. The configuration of the coils and pole pieces of the machine is such that, other things being

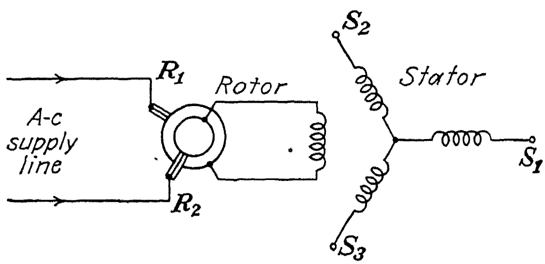

FIG. 2.7.—Synchro generator, schematic diagram.

equal, each of these stator voltages is proportional to the cosine of the angle between the magnetic axes of the rotor and the particular stator coil considered. If, therefore, the rotor position is varied at some constant rate by turning the rotor at constant speed, the three stator voltages, while being in phase or in phase opposition with each other, will be modulated sinusoidally at the rotation frequency of the rotor. This is shown by the curves of Fig. 2.8 for a rotor current frequency of 60 cycles per sec.. and a rotor speed of 3 r.p.s.

On the other hand, Fig. 2.9 shows the conditions for three typical *fixed* positions of the rotor (corresponding to positions A, B, and C in Fig. 2.8), referred to the stator coil S_1. The arrow which is placed near each coil represents the voltage induced in that coil at the instant at which the voltage in the rotor coil passes through its maximum positive value. In the case of Fig. 2.9A, the magnetic axes of the rotor coil and the stator coil S_1 are parallel to each other. The *turns ratio* of the rotor

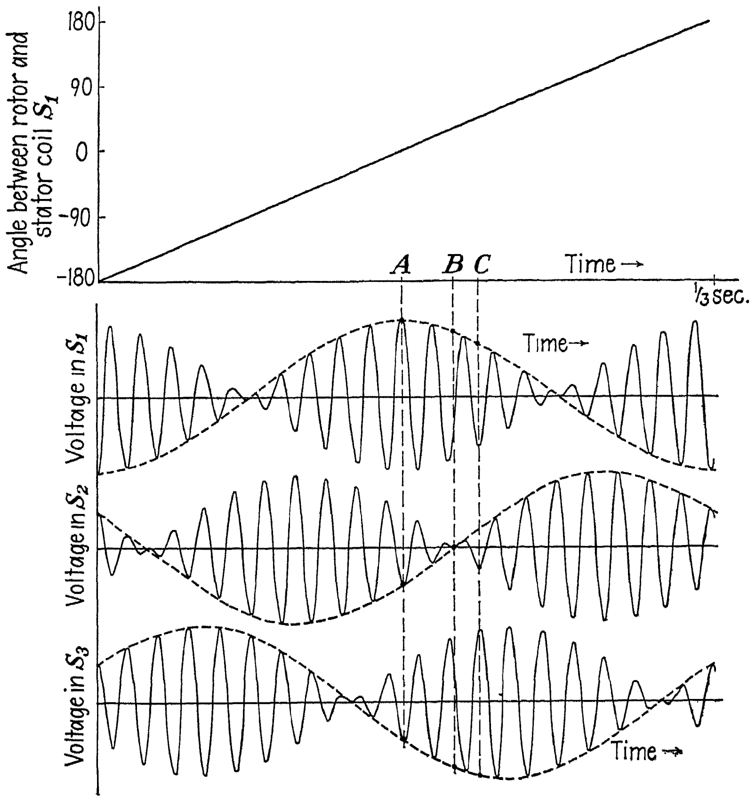

FIG. 2.8.—Synchro-generator stator voltages for constant-speed rotation of rotor

coil and any one of the stator coils is such that, under these conditions, a voltage of 115 volts applied to the rotor coil causes a voltage of 52 volts to appear across the stator coil S_1. Since the angle between any two stator coils is 120 deg., voltages of 52 cos 120° (or −26 volts) will then be induced in the coils S_2 and S_3. The positive and negative values represent here, of course, phase polarities of the voltages involved. Thus, at the instant that the *positive* voltage of 52 volts in the coil S_1 is directed from the center connection N toward the terminal S_1 of the coil, the *negative* voltages of −26 volts are directed from S_2 and S_3,

respectively, toward N. Referring now to Fig. 2.9B, the rotor is turned 30 deg. from its original position. The voltage across the coil S_1 is 45 volts (directed away from the common connection point of the three stator coils). The voltage across the coil S_3 is -45 volts (directed toward the stator common point). The voltage across S_2 is zero, since the magnetic axis of the rotor coil is perpendicular to that of S_2.

Finally, Fig. 2.9C shows the rotor coil rotated by 45 deg. Again multiplying the original voltage of 52 volts by the cosine of the angle between the magnetic axes of the rotor coil and each one of the stator coils, voltages of 36.8, 13.5, and -50.2 volts are found to appear across the stator coils S_1, S_2, and S_3, respectively.

Having thus described the operation of a synchro generator, it will be shown, in the following section, how the voltages developed in the generator stator coils allow the angular position of the generator rotor

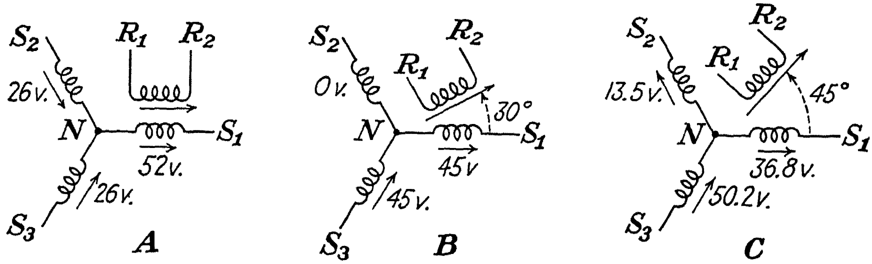

FIG. 2.9.—Synchro-generator stator voltages for three typical rotor positions.

shaft to be reproduced and followed by the shaft of a similar device designated as a *synchro motor*.

Synchro Motor.—The function of a synchro generator, as described above, consists essentially in producing a set of alternating voltages at its stator terminals, identifying the angular position of its rotor coil and shaft with respect to its stator coils. The function of a synchro motor is the counterpart of that of a synchro generator: the synchro motor receives the voltages produced by the synchro generator, and its rotor coil and shaft are thereby caused to orient themselves into the same angular position, when referred to the motor stator coils, as do the generator rotor coil and shaft, when the latter are referred to the generator stator coils. The construction of a synchro motor is substantially the same as that of a synchro generator, as illustrated in Fig. 2.6 and 2.7. However, a mechanical damper is usually mounted on the motor shaft to reduce the tendency of the latter to *overshoot* and oscillate in case of sudden speed variation of the synchro generator.

The inertia of the motor must, however, be kept to a minimum, in order that the motor may follow varying motions of the generator rotor

without undue time lag. The torque developed by usual synchro motor designs generally allows the motor to be loaded only lightly, if reasonable accuracy (of the order of 1 deg.) is desired in the position correspondence between the motor and generator rotor shafts.

Connections of a synchro motor to a synchro generator are made as shown in Fig. 2.10. As described before, the rotor coil terminals R_1 and R_2 of the generator are connected to an a-c supply line (say 115 volts, 60 cycles, for instance). The generator stator terminals S_1, S_2, and S_3 are connected to corresponding terminals S'_1, S'_2, and S'_3 of the motor stator coils. Finally, the rotor coil terminals R'_1 and R'_2 of the motor are connected, respectively, to the same a-c line as the corresponding rotor terminals R_1 and R_2 of the generator.

As explained, line-frequency (60-cycle) voltages are produced at the generator stator terminals S_1, S_2, and S_3, which are functions of the angular position of the generator rotor coil. These voltages, when

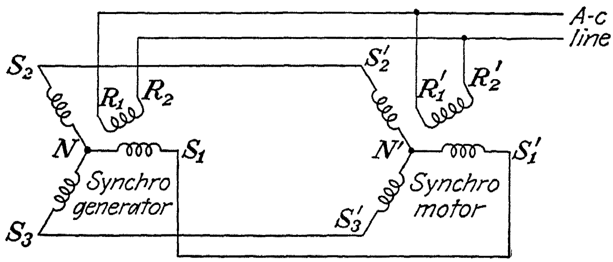

FIG. 2.10.—Interconnections between synchro generator and synchro motor.

applied to the motor stator terminals S'_1, S'_2, and S'_3, cause proportional alternating currents to flow in the motor stator coils. The magnetic fields thereby set up in these three coils combine into a resultant field that has the same orientation with respect to the motor stator coils as the generator rotor field with respect to the generator stator coils. If this resultant magnetic field in the motor were constant, a constant direct current in the rotor coil of the motor would cause the latter to orient itself so that its magnetic axis would coincide with that of the motor stator field. However, since the stator field is alternately reversed at the same frequency (60 cycles in the present instance) as the current in the generator rotor coil, the current in the motor coil must also be reversed in synchronism with this in order to provide a permanent orientation of the motor rotor coil. This is done here by connecting both the generator and motor rotor coils to the same a-c line. The orientation is then the same in the motor and generator.

Differential Synchro Generator.—The construction of a differential synchro generator, the function of which was described in a previous paragraph, differs from that of a simple synchro generator in that its

rotor has a three-coil distributed winding similar to that of the stator. Both the rotor and stator cores are therefore slotted, and the Y-connected rotor coils are brought out to three slip rings with which contact is made by means of brushes.

A typical connection of a differential synchro generator to a simple synchro generator is shown in Fig. 2.11. As before, the terminals R_1 and R_2 of the two-pole rotor of the synchro generator are connected to the a-c excitation line. The three generator stator terminals S_1, S_2, and S_3 are connected, respectively, to the three stator terminals S'_1, S'_2, and S'_3 of the differential generator. The three rotor terminals R'_1, R'_2, and R'_3 of the differential generator are connected to such terminal synchro repeater (motor or control transformer) as may be desired.

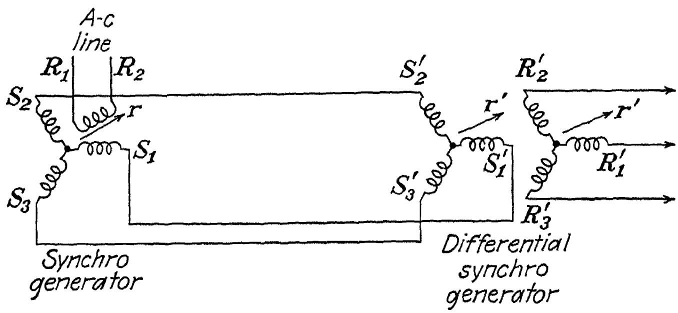

FIG. 2.11.—Synchro generator and differential synchro generator in corresponding positions.

It will be noted that the stator coils S'_1, S'_2, and S'_3 of the differential generator are connected to the stator coils S_1, S_2, and S_3 of the generator in exactly the same manner as the stator coils of a synchro motor were shown to be connected in the preceding section. It was stated in that preceding section that the magnetic field r set up by the rotor current in the generator would then be duplicated, in direction and proportional magnitude, in the stator coils S'_1, S'_2, and S'_3. This is true also in the present instance, as shown by the arrow r' in Fig. 2.11.

If now the rotor position of the differential synchro generator is such that the rotor coils R'_1, R'_2, and R'_3 are parallel to the corresponding stator coils S'_1, S'_2, and S'_3, these rotor coils and the magnetic field r' will be in the same relative position as the stator coils S_1, S_2, and S_3 and the field r in the generator. The voltages induced by the field r' in the coils R'_1, R'_2, and R'_3 are then equal (or proportional) respectively, to those induced by the field r in the coils S_1, S_2, and S_3. In other words, for this particular rotor position of the differential generator the output voltages of the latter are a repetition or duplication of the output voltages of the generator.

Suppose now that the rotor R_1-R_2 in the generator is kept stationary and that the rotor R'_1-R'_2-R'_3 in the differential generator is turned to some new position, say clockwise by an angle a, as shown in Fig. 2.12. It is obvious that the relative positions of the field r' and the rotor R'_1-R'_2,-R'_3 in the differential generator are now the same as if this rotor had been kept fixed in its original position and the rotor R_1-R_2 of the generator had been turned counterclockwise by the same (but opposite) angle a, since such a rotation is followed and duplicated by the field r' in the differential generator. The voltages produced at the rotor terminals R'_1, R'_2, and R'_3 of the differential synchro generator are thus the same as would be produced at the stator terminals of a simple synchro generator, the rotor of which would be turned from the *zero* position by an angle equal to the algebraic sum of the actual

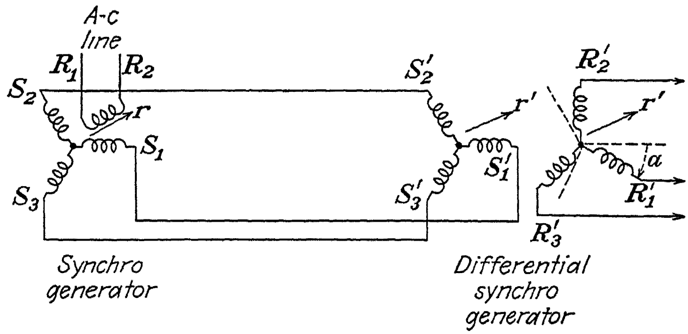

FIG. 2 12.—Synchro generator and differential synchro generator in differing positions.

positions of the shafts of the two machines. This allows a servo control system to be controlled from two different input points, since a given angular displacement of the shaft of either the generator or the differential generator will produce identical voltage variations at the output terminals R'_1, R'_2, and R'_3.

Differential Synchro Motor.—The construction of a differential synchro motor, as defined in a previous section, is substantially the same as that of a differential synchro generator, with the addition of a mechanical damper on the motor shaft. This damper serves the same purpose as in the case of a simple synchro motor.

Operating connections of a differential synchro motor are shown in Fig. 2.13. The output voltages (stator voltages) of two simple synchro generators A and B are applied, respectively, to the stator and rotor terminals of the differential synchro motor. The rotors R_1-R_2 and R''_1-R''_2 of the two generators are excited from a common a-c line. Under these conditions, the magnetic field r'_A in the stator coils of the motor will have the same angular position with respect to these stator

coils as the rotor field r_A with respect to the stator coils of the generator A. Similarly, the field r'_B in the rotor coils of the motor has the same angular position with respect to these rotor coils as the rotor field r_B has with respect to the stator coils of the generator B.

If then the rotors of the two generators are in the positions r_A and r_B shown in Fig. 2.13, the stator and rotor fields in the motor will have

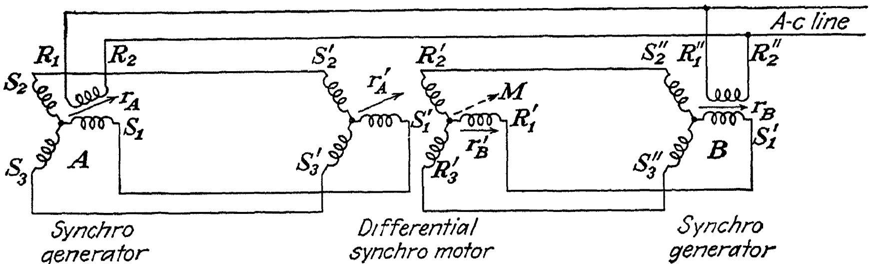

FIG. 2.13—Differential synchro motor positioning.

angular positions as shown by the arrows r'_A and r'_B, respectively. The rotor of the motor, being free to turn, will then rotate into a position M, which brings its field r'_B into coincidence with the stator field r'_A.

Suppose next that the rotor of the generator B is displaced counterclockwise as shown in Fig. 2.14. Since the rotor field r'_B of the differential motor duplicates this position with respect to the rotor coils of the motor, it will swing into the position shown in the figure. But as

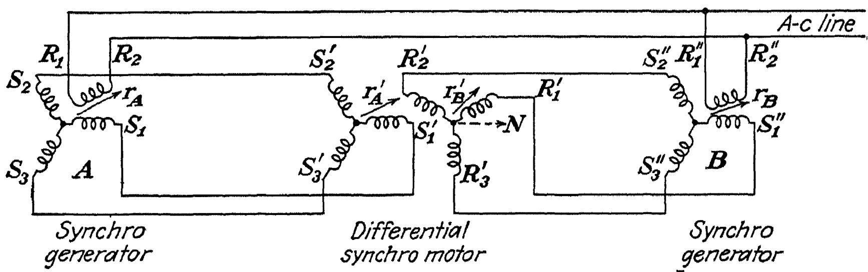

FIG. 2.14.—Differential synchro motor positioning.

the rotor of the motor is free to rotate, it will turn clockwise to a position N and thus bring the field r'_B into coincidence with the stator field r'_A.

The angular displacement of the motor shaft, therefore, is equal to the algebraic sum of the angular displacements of the two generator shafts.

Synchro Control Transformer.—A synchro control transformer is constructed in somewhat the same manner as a simple synchro generator. Its function was described in a previous section, and its operation is illustrated in Fig. 2.15.

The rotor R_1-R_2 of a synchro generator is excited, as usual, from an a-c line. The three stator terminals S_1, S_2, and S_3 of the generator are connected, respectively, to the corresponding stator terminals S'_1, S'_2, and S'_3 of the synchro control transformer. The rotor terminals R'_1 and R'_2 of the control transformer are generally connected to a high-impedance circuit, which may be the control grid circuit of a vacuum

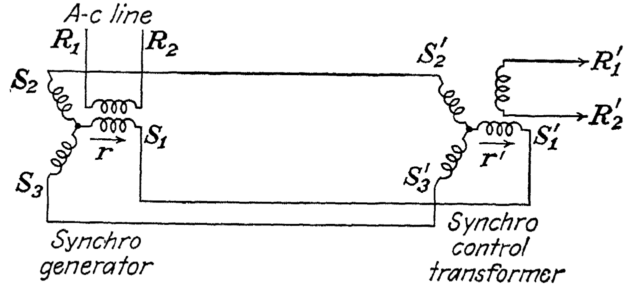

FIG. 2.15.—Synchro-control transformer operation.

tube amplifier, so that, for all practical purposes, negligibly small current will flow in the rotor coil. Since the output (rotor) voltage of the control transformer is to indicate the angular deviation of the generator rotor coil from some reference position, the magnetic axis of the transformer rotor coil must be set perpendicular to the field r' of the transformer stator coils when the rotor of the generator is in its reference position. This is the position shown in the diagram of Fig. 2.15.

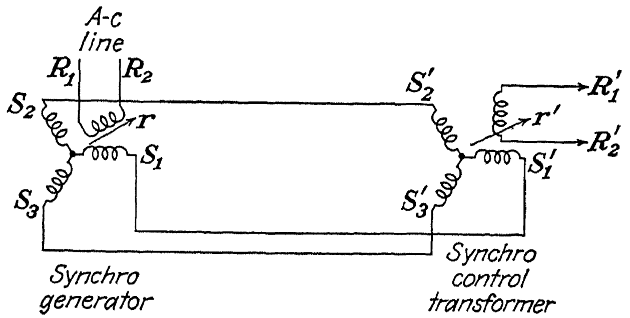

FIG. 2.16.—Synchro-control transformer operation.

If now the rotor of the generator is angularly displaced as shown in Fig. 2.16, the stator field r' in the control transformer will undergo similar displacement. This field will then cut the transformer rotor winding and will induce in the latter a voltage proportional to its projection along the magnetic axis of the rotor winding. Thus, the output voltage of the control transformer will be proportional to the sine of the angle of the generator rotor coil displacement.

In the usual types of control transformers, the *turns ratio* of the stator and rotor windings is such that the highest voltage induced in the rotor is 55 volts r.m.s. This corresponds to an angular displacement of 90 deg. from the *zero volt* position. The rotor voltage corresponding to any angular position θ, counted from the *zero volt* position, is thus $(55 \times \sin \theta)$ volts r.m.s. For an angle of 1 deg., the voltage is then

$$55 \times \sin 1^\circ = 55 \times 0.0175 = 0.96 \text{ volt}$$

or approximately 1 volt. Since for small angles the value of the sine can be considered as equal to that of the angle, the rotor voltage varies linearly with the angle about the zero position at the approximate rate of 1 volt per deg.

No appreciable current flows in the control transformer rotor coil, so no torque is developed, contrary to the conditions obtaining in a synchro motor. As will be seen in the following section, the rotor of the control transformer may be mechanically connected to the input member or to the output load of the servo control system. The operation of the controller then tends to *zero* the output voltage of the control transformer.

Typical Applications of Synchro Repeaters to Servo Control Systems. In the preceding chapter a simple block diagram of a servo control system was given (Fig. 1.10) to illustrate the operating relationship of the component elements of such a system. The use of electromechanical follow-up links, such as the self-synchronous repeaters described in the above sections, allows the various parts of the control system to be as widely separated from each other as may be required. Because each of the synchro units described has operating functions of its own, it is possible to combine them in various manners, according to the special requirements of the particular system to which they may be applied. The fundamental pattern of the system remains, of course, as described in the preceding chapter.

In general, the servo motor and the output load that it drives are located in the vicinity of each other and are mechanically interconnected either directly or by means of gears of suitable ratio. On the other hand, the location of the input member of the system, through which the controlling signal or intelligence is introduced into the system, depends on the particular conditions of installation of the equipment.

A possible arrangement is illustrated in Fig. 2.17. The servo motor is shown driving the output load through a reduction gear. The output shaft is also connected to the rotor of a synchro generator, the winding of which is excited from an a-c line. At the three stator terminals of the generator appear voltages that identify at every instant the angular position of the output load. These voltages are applied to the

corresponding stator windings of a synchro control transformer. The rotor of this transformer is mechanically connected to the input member, which is represented here by a handwheel attached to the transformer rotor shaft. The rotor output voltage of the transformer thus represents the angular difference (error) between the input member and output load. This voltage is applied to a suitable amplifier, which,

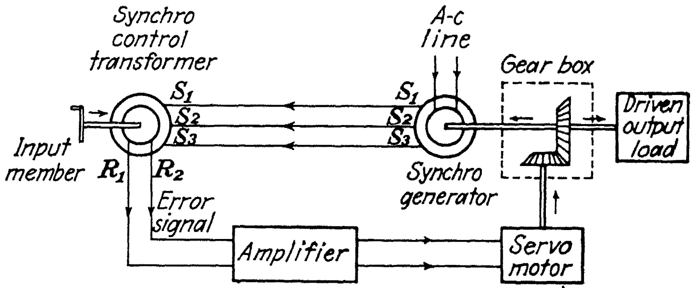

FIG. 2.17.—Simple servo system using synchro follow-up links.

in turn, feeds into the servo motor. The motor then drives the load and synchro generator in such direction as to cause the latter to follow the motion of the input member and thereby reduce to zero the output voltage of the control transformer.

By comparison with Fig. 1.10 of the preceding chapter, it is readily recognized that the synchro control transformer performs, in the present instance, the function of the differential element of the system as it

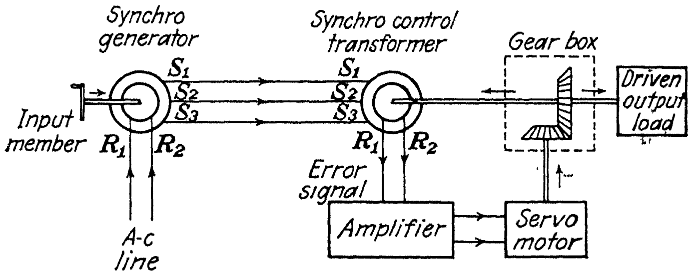

FIG. 2.18.—Another servo system using synchro follow-up links.

serves to *compare* the positions of the input and output members of the system. It need not be associated with any translating device, since its output is not a mechanical motion, but an electrical voltage, capable of operating the amplifier directly.

An equivalent system is obtained by interchanging the locations of the synchro generator and synchro control transformer. This is illustrated in Fig. 2.18, where it is the position of the load, instead of that of the input member, which determines the position of the rotor of the

control transformer. Conversely, it is the position of the input member, instead of that of the load, which sets the position of the synchro generator rotor and is then transmitted to the control transformer. The details of operation of this arrangement are sufficiently similar to those of the preceding example to require no further description.

The arrangement of Fig. 2.19 illustrates a possible application of a differential synchro generator. The system differs from the system of Fig. 2.18 in that the stator windings of the synchro generator are not connected to those of the synchro control transformer, but feed into the stator windings of a differential synchro generator. The rotor windings of this differential generator are connected to the stator terminals of

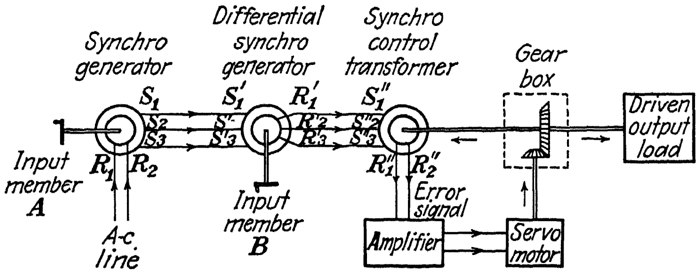

FIG. 2.19.—Servo system with two input members.

the synchro control transformer. Two input members are provided, represented by handwheels A and B connected mechanically to the rotor shafts of the synchro generator and differential synchro generator, respectively. As may be understood from Figs. 2.11 and 2.12, the displacement of either one of these input members produces an error voltage at the rotor output terminals of the synchro control transformer. This voltage is applied to the servo motor through the amplifier, and the motor drives the load and control transformer rotor in such a direction and by such an amount as to tend to zero the error voltage. If both input members A and B are displaced simultaneously, the error voltage developed by the control transformer and the resulting output load displacement are proportional and equal, respectively, to the algebraic sum of the displacements of the two input members.

CHAPTER III

FUNDAMENTALS OF MECHANICS AND ELECTRICITY

The servo-control systems discussed in this book involve a knowledge of both mechanical and electrical engineering principles. For the benefit of readers who may not be conversant with some of these principles, a brief review of the fundamentals of mechanics and electricity is therefore given below. Only the most elementary concepts are recalled here. More advanced principles are described in later chapters, as required for the setting up, discussion, and application of equations that express quantitatively the performance characteristics of complete control systems.

PRINCIPLES OF MECHANICS

Time and Space.—Time and space constitute the framework within which physical phenomena occur. In such engineering problems as will be discussed later, time is expressed in minutes or in seconds, while the dimensions of space will be expressed in yards, feet, or inches (1 yd. being equal to 3 ft., 1 ft. being equal to 12 in.).

Standard clocks, as well as standard lengths of 1 yd., are kept in a number of scientific institutions, and require no further discussion here.

Motion, Speed, and Acceleration.—Consider a fixed point A on a line $X'X$, Fig. 3.1. The *position* of any other point P of this line is defined by the distance x from the fixed reference point A to the point P, measured along the line $X'X$. It is expressed in any convenient unit of length.

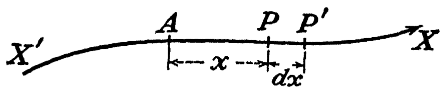

FIG. 3.1.—General motion of a point.

Motion of the point P along the line $X'X$, with respect to the reference point A, is a change of position of the point P. It is described by expressing the position of P as a function of time

$$x = f(t). \tag{3.1}$$

In this equation, $f(t)$ stands for the words *function of time* and implies that for any given value of the time t the corresponding value of the distance x is known. Actual calculation of x can be performed only when the nature of the function $f(t)$ is given, such as by stating, for example, that the value of x is proportional to the value of t, or to the square of the value of t, or more generally, when the method of calcu-

lating x in terms of t is specified. However, there are certain definitions and properties which apply irrespective of the particular method of calculation, and which can be discussed without explicitly stating the form or nature of the function $f(t)$.

Thus, the *speed* or *velocity* of the moving point P is defined as the rate of change of the position of P: if the point moves a distance dx from P to P' during a time interval dt, the speed v of the point P is

$$v = \frac{dx}{dt}. \tag{3.2}$$

If the time interval dt is infinitesimally small, the speed is therefore equal to the first time derivative of the function $f(t)$ in Eq. (3.1).

$$v = \frac{dx}{dt} = f'(t). \tag{3.3}$$

It is expressed in units of length per unit of time.

If the speed varies from one instant to another, its time rate of change, called the *acceleration* of the moving point P, is equal to the first time derivative of the speed; hence to the second time derivative of the function $f(t)$.

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = f''(t). \tag{3.4}$$

It is expressed in units of speed per unit of time, that is to say, in units of length per unit of time per unit of time, or in other words, units of length per unit of time squared.

For the purpose of description and measurement, the motion of a rigid (nondeformable) body is classified as either translatory or rotary motion, or a combination of these two types of motion.

Translatory motion is characterized by the fact that all points of the moving body travel along parallel and equal paths during any given time interval. The motion, speed, and acceleration of the body are then defined by the same equations and measured in the same units as just described for the case of the motion of a single point.

Rotary motion about a fixed axis is characterized by the fact that all points of the moving body travel through equal angles, measured about the axis, during any given time interval. The position of the body is then best expressed in terms of angle with respect to some original reference position or orientation. In this text, angles are preferably measured in radians, 1 radian being an angle that intercepts a circular arc equal to its radius. Since the circumference of a circle has a length equal to 2π radians (*i.e.*, 6.28 radians), 1 radian is equal to 57.3 deg.

Angular speed and acceleration are expressed, respectively, in units of angle per unit of time and in units of angle per unit of time per unit of time (*i.e.*, per unit of time squared). Angular speeds expressed in revolutions per unit of time are readily converted into radians per unit of time, one revolution being equal to 2π radians.

Figure 3.2 illustrates the rotary motion of a point P along the circumference of a circle of center O and radius r. The *position* of P is defined, with respect to a fixed point A of the circumference, by the *length* x of the arc AP, or by the *angle* $\theta = x/r$ subtended by this arc. If the point P moves from P to P' over an arc length dx during an elementary time interval dt, its *linear* speed and acceleration along the circle are equal, respectively, to dx/dt and d^2x/dt^2. Its *angular* speed and acceleration around the center O are equal to $d\theta/dt$ and $d^2\theta/dt^2$. In the case of rotary motion, therefore, it follows that

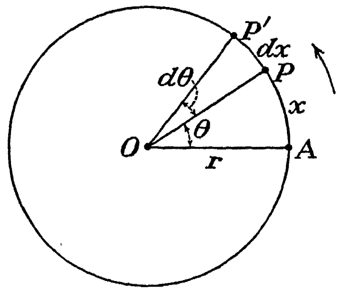

FIG. 3.2.—Rotary motion of a point.

$$\text{Linear speed} = r \times \text{angular speed} \tag{3.5}$$
$$\text{Linear acceleration} = r \times \text{angular acceleration.} \tag{3.6}$$

Weight and Mass.—A fundamental property of matter is the attraction exerted by any two material bodies upon each other. When one of the two bodies is the earth, the force of attraction on the other body is called the latter's *weight*. The force (or weight) is directly proportional to the quantity of matter, or *mass*, of the body.

It follows that equal masses have equal weights. The mass of a body can therefore be compared with that of another body (and *measured* if the latter is a standard) by means of a beam balance. On the other hand, measurement of the weight of a given body implies a measurement of the force of attraction of the earth on that body. The measurement can be made by means of a spring balance, the spring distortion being proportional to the magnitude of the force.

The unit of weight or of force is called the *pound*. It is equal to the weight, measured at sea level and a latitude of 45 deg., of a mass similar to that of a standard metal block that is kept at the U.S. Bureau of Standards.[1]

[1] In view of the slightly nonspherical shape of the earth, and because the gravitational force is inversely proportional to the square of the distance, the measured value depends on the geographical location. The maximum discrepancy between measurements made at the standard location and some other location is of the order of three parts in one thousand.

Translatory Motion, Force, and Mass.—Of itself, a piece of matter cannot change its state of motion or of rest: it is inanimate, or inert. This property, whereby a material body remains at rest or in a state of constant speed, straight line motion, until acted upon by some external agent, is called the *inertia* of the body. The external agent is called a *force* and is commonly described as a push or a pull. A force may change the direction or the speed of motion, or both, of the body to which it is applied. (If the body is prevented from moving, the force will distort its shape, as was mentioned in the preceding paragraph, in relation to the spring in a spring balance.)

If a body is allowed to move freely, a force applied to it will cause the body to move in the direction of the force. It has been found experimentally that the resulting acceleration (rate of change of the speed) is proportional to the applied force, and is inversely proportional to the mass of the moving body. Using suitable units, this may be expressed

$$\text{Acceleration} = \frac{\text{force}}{\text{mass}} \tag{3.7}$$

This can also be written

$$\text{Force} = \text{mass} \times \text{acceleration}$$

or, in abbreviated form,

$$F = Ma. \tag{3.8}$$

Thus, a unit force applied to a body having a unit mass produces a unit acceleration. In particular, a force of 1 lb. will impart an acceleration of 1 ft. per sec. to a body having a unit mass. This unit mass is called a *slug*.

A body falling freely toward the earth moves under the action of its weight W, which is represented by the force F in the above Eq. (3.8). The acceleration due to this gravitational force is, as determined experimentally, equal to 32.2 ft. per sec. It is generally represented by the letter g, instead of the letter a used in Eq. (3.8). This equation, when applied to a freely falling body, can therefore be rewritten

$$\text{Weight} = \text{mass} \times g$$

or in abbreviated form

$$W = Mg \tag{3.9}$$

or

$$W = 32.2M. \tag{3.10}$$

Thus, the weight of a mass of 1 slug is equal to 32.2 lb.

In the metric (c.g.s.) system the dyne is a force that, applied to a mass of 1 gm., imparts to it an acceleration of 1 cm. per sec. per sec. The gram and centimeter are subdivisions of the standard units of mass

(kilogram) and length (meter); prototype copies of which are kept at the U.S. Bureau of Standards.

Rotary Motion, Torque, and Moment of Inertia.—In the preceding paragraphs, translatory motion was described in terms of mass, force, and acceleration. In the case of rotary motion corresponding but different concepts must be used. This may be understood by first com-

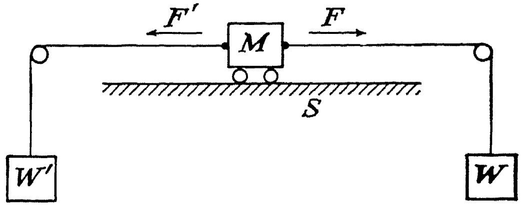

FIG. 3.3.—Equilibrium conditions of translatory system.

paring the equilibrium conditions of two systems capable, respectively, of translatory and rotary motions.

Consider, for example, a body of mass M, Fig. 3.3, supported by rollers on a horizontal surface S. Two forces F and F', parallel to the surface, are applied to the body in opposite directions. This may be done by attaching two weights W and W' to the body through flexible cords passing over two fixed pulleys mounted on each side of the body. The forces F and F' are then equal, respectively, to the weights W and

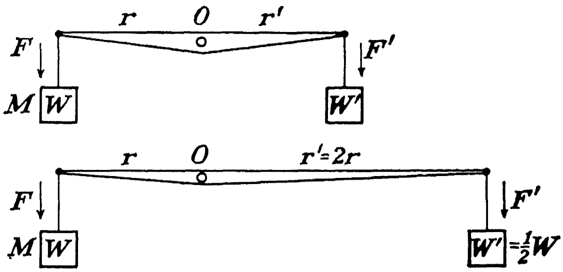

FIG 3.4.—Equilibrium conditions of rotary system.

W'. If the forces are of different magnitudes, the body M will move with increasing speed in the direction of the greater of the two forces; and the motion will be the same as though only a single force, equal to the difference of F and F', were applied to the body. If this difference is zero, the state of motion or of rest of the body will remain unchanged. Thus, in the case of translatory motion only the *magnitude* and *direction* of the applied forces enter into play.

The case of rotary motion is illustrated in Fig. 3.4. A beam may rotate in a vertical plane around a horizontal pivot O. A body of mass M and weight W is attached to one end of the beam at a distance r from

the pivot. To the other end of the beam a weight W' is attached at a distance r' from the pivot. The system is in equilibrium if the two weights W and W' (*i.e.*, the forces applied to the ends of the beam) *and* the two arm lengths r and r' are equal. But if the length r' of the arm that supports the weight W' is, for example, twice the length r of the arm that supports the weight W, then the weight W' must be equal to one-half the weight W if the system is to remain stationary in the same equilibrium position.

Generally, the condition of equilibrium or of motion of a system capable of rotary motion is not determined simply by the magnitudes and directions of the forces applied to it, but by the products of these forces and the distances of the pivot to their respective lines of action. In the present example, the effectiveness of the force W is measured by the product rW, while that of the force W' is expressed by the product $r'W'$. Such a product is called the *moment* or *torque* of the particular force considered. Being the product of a length by a force, it is expressed in *foot-pounds* or *inch-ounces*, according to what units of length and force are used.

Consider now, Fig. 3.5, a body of mass M and small linear dimensions, free to rotate about a pivot O, along a circle of radius r. Applied to the body is a force F of constant magnitude, lying in the plane of the circle described by M, and directed at all times perpendicularly to the instantaneous direction of the radius OM. Under these conditions, the body will move along the circle at constantly increasing speed. Let MM' represent the path covered by the body during a given time interval. If this interval is sufficiently small, this circular path MM' may be likened to a small straight line segment along the direction of the force F. The mass M, force F, and linear acceleration a along MM' are then tied by the relation

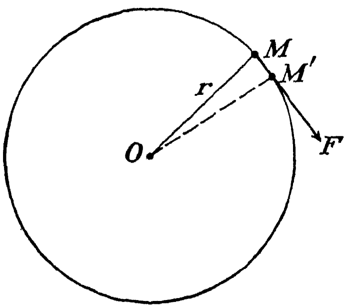

FIG. 3.5.—Rotary motion of a small mass.

$$F = Ma \tag{3.8}$$

found before in the case of translatory motion. Replacing the linear acceleration a by the corresponding angular acceleration α, as defined in a previous paragraph this expression becomes

$$F = Mr\alpha. \tag{3.11}$$

Moreover, in the case of rotary motion, it is not the force F, but its moment or torque rF that defines its effectiveness on the system. Thus,

multiplying both members of the last expression by r, this becomes

$$rF = Mr^2\alpha. \tag{3.12}$$

In this expression, α is the angular acceleration of the mass M, the product rF is the torque or moment of the force F, and the product Mr^2 is defined as the *moment of inertia* of the small mass M. The relation is thus obtained:

Torque = moment of inertia × angular acceleration.

Writing this in abbreviated form

$$T = J\alpha, \tag{3.13}$$

and comparing it with the expression

$$F = Ma \tag{3.8}$$

previously found for translatory motion, it is seen that the force, mass, and acceleration in the latter expression are replaced, respectively, by torque, moment of inertia, and angular acceleration, in the preceding expression of rotary motion.

The moment of inertia, being the product of a mass by a length squared, is expressed in corresponding units, such as slug-ft.², for example.

The preceding discussion relates to the case of a rotating body, the linear dimensions of which are *small* with respect to the radius of the circular trajectory. If this condition is not fulfilled, the various parts of the rotating body, while having equal *angular* speeds and accelerations, will have *linear* speeds and accelerations which differ according to their distances to the center of rotation. However, if the body is thought of as divided into small elements of mass m, the moment of inertia of each element is equal to the product mr^2 of its mass by the square of its distance r to the axis of rotation, as found above. The moment of inertia of the entire body about the axis of rotation is then the sum of the moments of inertia of its elementary component parts.

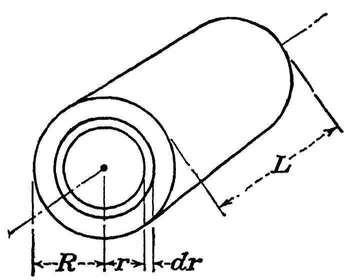

FIG. 3.6.—Moment of inertia of rotating cylinder.

$$J = \Sigma mr^2, \tag{3.14}$$

or, if these parts are infinitesimally small mass elements dm,

$$J = \int r^2\, dm. \tag{3.15}$$

For simple configurations of the rotating body, this may be calculated readily from elementary geometrical considerations. For example,

consider (Fig. 3.6) a cylinder of length L, radius R, and mass M rotating about its axis. The volume of this cylinder is

$$V = \pi R^2 L \tag{3.16}$$

and the density (mass per unit volume) of the cylinder material is

$$D = \frac{M}{\pi R^2 L} \tag{3.17}$$

From the preceding reasoning it is seen that if the cylinder is thought of as constituted by a large number of small elementary masses, those elementary masses which are at the same distance from the axis will all have the same moment of inertia. Let then r be the radius of an annular shell within the cylinder. The circumference of this shell is $2\pi r$. If the shell has an infinitesimally small thickness dr, the area of its annular base is $2\pi r\, dr$, and the volume of the shell is $2\pi rL\, dr$.

The mass dm of the shell is then

$$\begin{aligned} dm = D2\pi rL\, dr &= \frac{M}{\pi R^2 L} 2\pi rL\, dr \\ &= \frac{2M}{R^2} r\, dr. \end{aligned} \tag{3.18}$$

If the radius r is made to vary between zero and the radius R of the cylinder, the moment of inertia J of the cylinder is, according to Eq. (3.15)

$$J = \int_0^R r^2\, dm = \int_0^R r^2 \left(\frac{2M}{R^2} r\, dr\right) = \frac{2M}{R^2} \int_0^R r^3\, dr \tag{3.19}$$

or

$$J = \frac{2M}{R^2} \frac{R^4}{4} = \frac{MR^2}{2} \tag{3.20}$$

Similarly, for a hollow cylinder of mass M, outer radius R, and inner radius R' rotating about its axis, the moment of inertia is found to be

$$J = \frac{(R^2 + R'^2)M}{2} \tag{3.21}$$

The *center of gyration* of a rotating body with respect to the axis of rotation is a point located at such a distance from the axis that, if the entire mass of the body were concentrated at that point, its moment of inertia would be the same as that of the body considered. The *radius of gyration* of the body is the distance from the axis of rotation to the center of gyration. From the preceding discussion the radius of gyration is seen to be equal to $\sqrt{J/M}$.

Friction.—In addition to forces applied externally to a body to alter its state of motion, certain forces arise as a result of the contact between

the moving body and stationary surfaces of other bodies touching or supporting it. To illustrate this point, consider a body A, Fig. 3.7, of weight W, resting on a flat horizontal surface S. It is found experimentally that it is necessary to apply a constant driving force D to the body in order to keep the latter in constant speed motion. Now, according to Eq. (3.8), application of a constant force to a body should produce a constant acceleration of the body. The constant-speed motion obtained in the present instance (zero acceleration) therefore indicates that, in addition to the driving force D, an equal, oppositely directed retarding force F operates on the body.

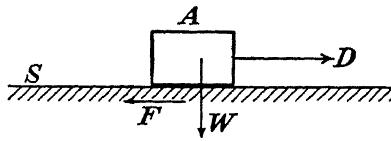

FIG. 3.7.—Friction of a moving body.

This force F is defined as the *friction force* between the body A and the supporting surface S along which it is moving. The forces F and D counterbalance each other exactly, so the net force acting on the moving body is zero, and its constant-speed motion is then in agreement with the conditions of Eq. (3.8).

Depending on the nature of the substances constituting the moving and stationary bodies and the condition of the contact surfaces between them, the friction force may or may not be a function of the contact pressure and of the relative speed of the two bodies.

Viscous friction is defined as a condition in which the friction force F is directly proportional to the speed v. For translatory motion, this is expressed by the relation

$$F = fv \tag{3.22}$$

where the constant of proportionality f is called the *friction coefficient.*

For rotary motion, the expression becomes

$$T = f\omega \tag{3.23}$$

where T is the friction torque, f a new friction coefficient, and ω the angular velocity.

Coulomb friction is a condition where the friction force is independent of the speed.

Still other types of friction arise in practice, where the friction force is proportional to some higher power of the speed, such as during the high-speed motion of a solid body through a fluid (liquid or gas).

Static friction may also be present between two solid bodies at rest, when a certain minimum force must be applied in order to start the motion of one body against the other.

In most of the applications discussed in this book, only viscous friction will be considered.

Work, Power, and Energy.—When the point of application of a force is allowed to move, the product of its linear displacement by the component of the force along the path of the point is called the *work of the force.* If s represents the displacement and F the component of the force, the work U is expressed

$$U = Fs. \tag{3.24}$$

Units of work are the foot-pound or, more generally, a combination of units of length and force. Similarly, if a torque T is applied to a body undergoing an angular displacement θ, the *work of the torque* is defined as the *product*

$$U = T\theta. \tag{3.25}$$

Power is the rate of doing work. It is thus equal to the first time derivative of work, and is expressed

$$P = \frac{dU}{dt} = \frac{dFs}{dt} = F\frac{ds}{dt} = Fv \tag{3.26}$$

or

$$P = \frac{dU}{dt} = \frac{dT\theta}{dt} = T\frac{d\theta}{dt} = T\omega \tag{3.27}$$

for linear and angular displacements, respectively, in the cases of constant applied force or torque. The linear and angular velocities are expressed by the letters v and ω, respectively, in these relations.

Power is expressed in *units of work per unit of time,* for example in foot-pounds per second. Other units frequently used are the *horsepower,* which is equal to 550 ft.-lb. per sec.; and the *watt* (more often used for measuring electrical power, as will be seen later), which is equal to $\frac{1}{746}$ part of a horsepower.

According to Eq. (3.27), the following relation can then be written

Power in ft.-lb. per sec. = torque in ft.-lb. × angular velocity in radians per sec. (3.28)

Using other units, this same relation is written

Power in hp. = torque in in.-oz. × angular velocity in r.p.m. × 10^{-6}, (3.29)

which is accurate within less than 1 per cent.

Energy is defined as an ability or capacity to do work. It may assume a variety of forms, such as mechanical, electrical, chemical, luminous, or calorific. Mechanical energy can, in turn, be either potential or kinetic, as determined, respectively, by the relative position or motion of material bodies. It is the transformation process of energy

from one of its forms into another that constitutes what is designated as *work*. The total amount of energy of a system, regardless of form, therefore remains constant. These statements are illustrated below for the case of mechanical energy.

Potential Energy.—When a spring is compressed or stretched, it is capable of doing work as it is released and returns to an unloaded condition. The energy that is thereby expended was stored in the compressed or stretched spring in a latent or potential form. Losses due to friction or other factors being disregarded, this energy is equal to that which was spent when the spring was originally compressed or stretched.

Thus, consider a spring in which the force F is proportional to the elongation x, or departure from the unloaded spring length

$$F = Kx \tag{3.30}$$

where K is a constant called the *spring constant.*

The unloaded spring is shown as AB in Fig. 3.8. Suppose that one of its ends A is stationary while the spring is being stretched by an amount X to bring its other end B into the position B'. For any intermediate point P between B and B', corresponding to an elongation x, the spring force can be calculated from the above Eq. (3.30). The work done in displacing the spring end from P to a point P' over an infinitesimal distance dx (so small that the force variation over this elementary elongation can be neglected) is equal to the product $F\,dx$. Substituting for F the value given by Eq. (3.30), this product becomes

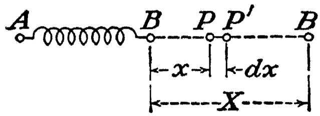

FIG. 3.8.—Potential energy stored in a spring.

$$F\,dx = Kx\,dx. \tag{3.31}$$

The total energy expended, or work done in stretching the spring by the amount X by displacing the spring end from B to B', is then

$$U_p = \int_0^X Kx\,dx = \tfrac{1}{2}KX^2. \tag{3.32}$$

This expression represents the potential energy stored in the elongated spring. This energy becomes available for performing work when the spring is released.

Kinetic Energy.—Consider a constant force F applied to a mass M, to which it imparts a constant acceleration a, these quantities being tied by the relation previously encountered

$$F = Ma. \tag{3.8}$$

Let v be the speed of the moving mass at some instant of time t. During the following infinitesimal time interval dt, the mass will travel a distance

equal to $v\,dt$, and the work done by the force during this same time interval is equal to the product $Fv\,dt$.

Substituting for F the value given by Eq. (3.8), this product becomes

$$Fv\,dt = Mav\,dt \tag{3.33}$$

and since the acceleration is equal to the time derivative dv/dt of the speed, this can be written

$$Fv\,dt = Mv\frac{dv}{dt}\,dt \tag{3.34}$$

$$Fv\,dt = Mv\,dv. \tag{3.35}$$

The work done, or energy expended, in accelerating the mass from rest (zero speed) to some speed value V is then equal to

$$U_k = \int_0^V Mv\,dv = \tfrac{1}{2}MV^2. \tag{3.36}$$

This so-called *kinetic energy* is associated with, or stored in, the moving mass.

Oscillatory System.—The behavior of a system capable of storing both potential and kinetic energy is closely related to certain phenomena encountered in servo control systems. A typical example will therefore be discussed here, in view of applications described in later parts of this book. Consider a mass M resting on a flat horizontal surface, Fig. 3.9, and attached to two fixed anchor points G and H through identical springs S and S'. As shown in the upper sketch, the position of equilibrium or of rest of the system corresponds to a position of the mass halfway between the anchor points.

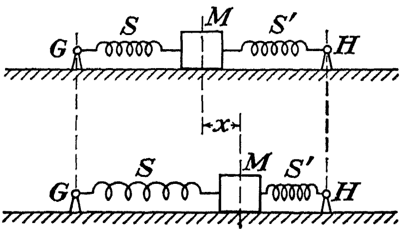

FIG. 3.9.—Oscillatory system.

Suppose that the mass is now displaced, by some external means, and suddenly released. It is a common experience that the forces exerted by the springs will drive the mass back toward its original central position. The mass will settle in this position either gradually or after a few oscillations about it, depending on the amount of friction between the mass and its supporting surface. The displacement of the mass being measured from the center, let then the following symbols be used:

x = displacement of the mass from its central position
v = velocity of the mass
M = magnitude of the mass
K = spring force per unit displacement
f = friction coefficient (only viscous friction is considered here).

During any short time interval dt, the kinetic energy variation of the system is

$$d(\tfrac{1}{2}Mv^2) = Mv\,dv, \tag{3.37}$$

while the potential energy variation is

$$d(\tfrac{1}{2}Kx^2) = Kx\,dx; \tag{3.38}$$

and the energy lost in the friction of the system is equal to the work of the friction force. This is expressed by the product of the friction force fv by the displacement $v\,dt$ of the mass during the time interval dt.

$$fv \cdot v\,dt = fv^2\,dt. \tag{3.39}$$

Since the total energy of the system remains constant, the algebraic sum of these energy variations is zero.

$$Mv\,dv + Kx\,dx + fv^2\,dt = 0, \tag{3.40}$$

or, dividing through by dt,

$$Mv\frac{dv}{dt} + Kx\frac{dx}{dt} + fv^2 = 0. \tag{3.41}$$

In order to express the displacement x and velocity v as functions of time, this equation must be transformed into equations containing x only and v only, respectively. This can be done by using the relation

$$v = \frac{dx}{dt}, \tag{3.2}$$

which simply states that the velocity, as defined in the early part of this chapter, is the time rate of change of the displacement.

Thus, substituting dx/dt for v in Eq. (3.41), this becomes

$$M\frac{dx}{dt}\frac{d^2x}{dt^2} + Kx\frac{dx}{dt} + f\frac{dx}{dt}\frac{dx}{dt} = 0, \tag{3.42}$$

or, dividing through by dx/dt,

$$M\frac{d^2x}{dt^2} + Kx + f\frac{dx}{dt} = 0. \tag{3.43}$$

Second-order differential equations of this same form will be encountered in later chapters relating to the operation of servomechanisms. Its solution will be found to be expressible as an exponential function of time

$$x = X\epsilon^{pt}. \tag{3.44}$$

In general, for small values of the friction, this is a complex function of time, which represents a damped oscillation and may be written in the

form

$$x = \epsilon^{-at}[X_1 \cos bt + X_2 \sin bt]. \tag{3.45}$$

In this equation, a defines the rate of decay of the oscillation, and b is the radian frequency of oscillation. The motion whereby the mass returns to its position of equilibrium has a marked similarity with the motion of a servo system, as will be brought out in the chapters that follow.

PRINCIPLES OF ELECTRICITY

Electricity and Matter.—Electricity is an essential element of the physical structure of matter. In the following brief discussion, only an oversimplified explanation of certain electrical phenomena will be given, which, however, will suffice for the limited purpose of this chapter. Within this restricted scope each atom of a material substance may be considered as an assemblage of minute particles, or "charges" of electricity. These charges are of two kinds designated, respectively, as *positive* and *negative*. Unlike charges attract each other, but are kept in balanced equilibrium by the internal forces of each atom. When a piece of matter is in an electrically "neutral" condition, the sum of the positive charges is equal to that of the negative charges within each of its atoms, and no external electric action is then apparent.

The elementary negative charges, also called *electrons*, are all identical with each other. In certain substances, particularly the metals, some of the electrons are but loosely tied to any given atom, and may pass from one atom to another. Such electrons are therefore called *free electrons*.

Electromotive Force and Electric Current Flow.—There are many ways of bringing forth electrical properties of matter. An example is an electric battery, which may be composed of a strip of zinc and a strip of copper partly immersed in a jar containing a slightly acid solution. A reaction between the battery components electrically polarizes the battery, the copper strip becoming positive and the zinc strip negative. The copper and zinc external terminals of the battery thereby become capable, respectively, of exerting attracting and repelling forces on negative charges of electricity, such as the free electrons within a piece of metal. The battery is thus a source of electromotive force.

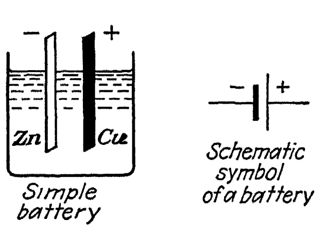

FIG. 3.10.—Electric battery and its schematic symbol.

In order to apply this electromotive force to the free electrons within a metal wire, it is simply necessary to connect the ends of the wire to the two battery terminals. The free electrons then move along the wire from the negative (zinc) terminal of the battery to the positive (copper)

terminal, then through the liquid within the battery[1] to the zinc strip, whence they again move along the wire from the negative to the positive terminal. The (negative) electrons do not accumulate on the copper strip, because it is a property of the battery to maintain the copper terminal positive with respect to the zinc terminal by a certain amount. The process thus constitutes a continuous flow or current of electricity around the closed circuit formed by the wire and battery.[2] The rate of flow, or intensity, of this current is measured by the quantity of electricity (number of electrons) passing through any one cross section of the wire during a unit of time. The unit of quantity of electricity, or of electric charge, is called the *coulomb* and is equivalent to the charge of approximately 6.28×10^{18} electrons. The unit of current intensity is the ampere. A current of 1 amp. carries a charge of 1 coulomb through a cross section of the wire during every 1-sec. time interval.

Electric Resistance.—As in the case of a mechanical force acting on a material body, an electromotive force accelerates such charges or free electrons to which it may be applied. However, in the preceding instance of current flow in a metal wire, the free electrons in the course of their motion enter into frequent collision with the atoms of the wire that happen to be in their path. This tends to reduce the speed of the electrons, as though a retarding force were applied to them. A constant average speed is then reached when the average retarding force is equal to the driving, or applied, electromotive force. Now, the number of collisions per unit of time and the corresponding average retarding force are proportional to the electron speed; hence to the current intensity in the wire. The current intensity i and applied electromotive force e are thus proportional to each other, which may be written

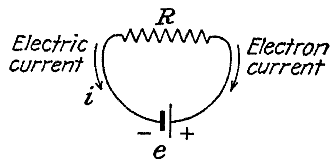

FIG. 3.11.—Current flow in a resistance.

$$e = iR \tag{3.46}$$

where the proportionality constant R is called the *resistance* of the wire. This equation is known under the name of *Ohm's law.*

Units of electromotive force and of resistance are called the *volt* and *ohm*, respectively. Their relation to the units previously defined, by which an electromotive force of 1 volt applied across a resistance of 1 ohm causes a current of 1 amp. to flow, may be established as follows:

[1] The process is actually more complicated than here described, involving the formation and motion of positive and negative ions through the liquid of the battery.

[2] Before the application of the electron theory, electric current was arbitrarily defined as flowing through the wire from the positive to the negative terminal of the wire. Thus, the electron current and electric current are seen to flow in opposite directions.

Work is performed by the electromotive force applied to the charges that through their motion constitute the electric current. If e is the electromotive force applied to the ends of a wire that carries current, the power expended, or work per second, is equal to the product

$$w = \frac{dq}{dt} e \tag{3.47}$$

where dq/dt is the time rate at which electric charge passes through any given cross section of the wire. Applying the definition of current intensity, as given above, this expression can be written

$$w = ei \tag{3.48}$$

where e is expressed in volts, i in amperes, and w in watts. One watt is equal to 0.737 ft.-lb. per sec., or, in equivalent manner, 746 watts are equal to 1 hp.

Substituting Eq. (3.46) in the relation, Eq. (3.48), this becomes

$$w = i^2R = \frac{e^2}{R} \tag{3.49}$$

where R is the resistance of the wire, i the current through the wire, and e the applied electromotive force.

The energy thus spent during every second is transformed into heat and raises the temperature of the wire. This property is used in electric heating applications, and also, as a special case, in the common electric incandescent light bulb.

Voltage-current Relation in a Resistance.—Ohm's law, as expressed by Eq. (3.46), shows that if either the electromotive force (or voltage) or the current i is varied, the other of these two quantities varies simultaneously in the same proportion. If, therefore, e or i is some function of time $f(t)$, the other is equal to this same time function multiplied or divided by the constant factor R. Both current and voltage can then be represented graphically in function of time by curves of similar shapes. The current and voltage may even be represented by one and the same curve, if the scales of ordinates are properly chosen.

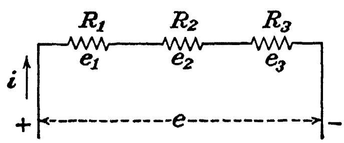

FIG. 3.12.—Series-connected resistances.

The currents and voltages operating in a circuit composed of a plurality of pure resistance elements are then expressed in very simple manner, irrespective of the connection arrangement of these elements and of the time dependence of the current or voltage.

Thus, if a circuit comprises several resistance elements R_1, R_2, R_3 . . . in series with each other, a current of intensity i flowing in the circuit will

set up voltages e_1, e_2, e_3 . . . across these respective elements equal to

$$e_1 = iR_1 \qquad e_2 = iR_2 \qquad e_3 = iR_3 \qquad \cdots$$

so that the over-all voltage across the entire circuit is

$$\begin{aligned} e &= e_1 + e_2 + e_3 + \cdots \\ &= iR_1 + iR_2 + iR_3 + \cdots \\ &= i(R_1 + R_2 + R_3 + \cdots) = iR \end{aligned} \tag{3.50}$$

where R is the sum of the component resistances of the circuit.[1]

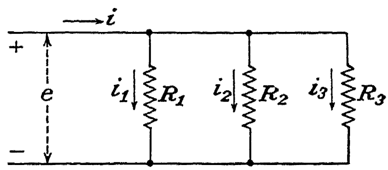

FIG. 3.13.—Parallel-connected resistances.

Similarly, if the circuit is composed of parallel-connected resistance elements R_1, R_2, R_3 . . . across which a voltage e is applied, the currents i_1, i_2, i_3 . . . in the individual elements are equal to

$$i_1 = \frac{e}{R_1} \qquad i_2 = \frac{e}{R_2} \qquad i_3 = \frac{e}{R_3} \qquad \cdots$$

The total current flowing through the circuit terminals is then

$$\begin{aligned} i &= i_1 + i_2 + i_3 + \cdots \\ &= \frac{e}{R_1} + \frac{e}{R_2} + \frac{e}{R_3} + \cdots \\ &= e\left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots\right) = \frac{e}{R} \end{aligned} \tag{3.51}$$

where

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots \tag{3.52}$$

Capacitance, or Electrostatic Capacity.—Suppose that the two terminals of a battery, instead of being connected to each other by an external wire as in the preceding discussion, are connected, respectively, to two metal plates A and A', Fig. 3.14. Let these plates be parallel to and facing each other, but separated by a small distance, to constitute what is called a *condenser* or *capacitor*. Some of the free negative electrons in the metal of the plate A' connected to the positive terminal of the battery are then attracted toward the latter and removed from the plate. How-

[1] These equations illustrate the operation of voltage dividers: by connecting several resistances R_1, R_2, R_3, . . . in series across a supply voltage e, voltages equal to a fraction of this supply voltage may be taken off the terminals of any one of these resistances. Thus, for example,

$$\frac{e_1}{e} = \frac{R_1}{R_1 + R_2 + R_3 + \cdots} = \frac{R_1}{R}.$$

ever, they do not accumulate on the positive electrode of the battery, for the electromotive force of the battery drives them through the battery to the negative electrode and from there through the connecting wire onto the plate A.

The resulting deficiency of electrons on the plate A' and excess of electrons on the plate A cause these plates, respectively, to become charged positively and negatively. If under these conditions the battery were now removed from the circuit and the capacitor plates connected to each other through a metal wire, the electrons accumulated on the plate A would flow back onto the plate A', canceling the electric charge unbalance in the system and restoring the original electric equilibrium. Thus, charging of the capacitor by the battery creates an electric force across the capacitor, which tends to oppose that of the battery. The charging process described above ceases when these two forces are equal and balance each other.

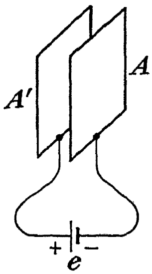

FIG. 3.14.—Capacitor circuit.

The charge q on the capacitor plates is proportional to the voltage e across the latter. This may be written

$$q = Ce. \tag{3.53}$$

The proportionality constant C is called the *capacitance*, or *electrostatic capacity*, of the condenser. It is expressed in farads, 1 farad being the capacitance of a condenser in which a charge of 1 coulomb corresponds to a voltage of 1 volt. Subdivisions of this unit, which are frequently encountered, are the microfarad and the micromicrofarad, which are, respectively, the millionth part (10^{-6}) and millionth of a millionth part (10^{-12}) of a farad.

Equation (3.53) relates the charge on the capacitor plates to the voltage across these plates. When this voltage is constant, the charge is also constant, and there is no motion of electricity in the system. Suppose, however, that instead of connecting the capacitor across a battery or other source of constant electromotive force, it is connected across a generator, the terminal voltage of which is being increased at a constant rate by a fixed number of volts during every second. In order that the capacitor voltage e and charge q may at every instant remain tied by the above equation (3.53), the capacitor charge must then increase accordingly.

In other words, a same number of electrons are transferred during every second from the plate A' to the plate A through the connecting wires and the generator, creating a constant flow or current of electricity along the circuit. The *intensity* of this current increases and decreases with the *rate of change* of the voltage across the capacitor plates. The

direction of current flow reverses with the *direction of variation* of this voltage. Thus, if i is the current intensity, as measured by the charge passing during every second through a cross section of the wire, while dq/dt and de/dt are, respectively, the time variation rates of the capacitor charge and voltage, the relation may be written

$$i = \frac{dq}{dt} = C\frac{de}{dt}. \tag{3.54}$$

A voltage variation of 1 volt per sec. across a capacitance of 1 farad produces a current flow of 1 amp. in the circuit.

Inductance, and Electromagnetic Induction.—Consider a permanent magnet, M, Fig. 3.15, producing a magnetic flux ϕ between its pole faces, and let A represent a metal wire bent into a circular loop enclosing this magnetic flux. It is found experimentally that if the loop is displaced in the direction D, for example, so that it will link with a gradually diminishing portion of the magnetic flux, an electromotive force will be set up, or *induced*, in the loop, as shown by the arrow e in the figure. Expressing the flux in suitable units, this induced electromotive force is equal to the time rate of change of the flux in the loop, both in magnitude and in polarity. This may be written

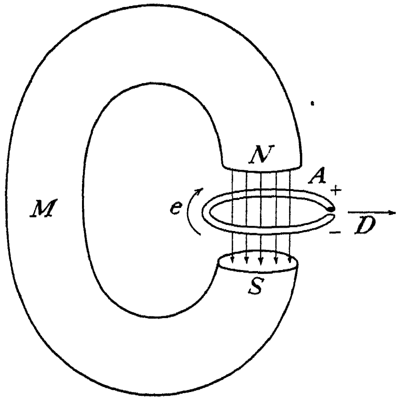

FIG. 3.15.—Electromagnetic induction.

$$e = \frac{d\phi}{dt}. \tag{3.55}$$

Such an induced electromotive force is observed, irrespective of the manner in which the magnetic flux and its variation are produced. In this connection, another property may be recalled, whereby a wire AA',

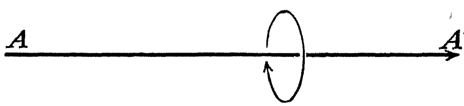

FIG. 3.16.—Magnetic field about a current-carrying wire.

Fig. 3.16, carrying an electric current flowing from A to A', is surrounded by a magnetic field. This can be shown by placing a small pivoted magnet, a compass needle for example, in the vicinity of the wire. The needle then assumes a direction perpendicular to the plane containing its pivot and the wire. The magnetic field may be mapped out by moving the needle in the direction in which it points and noting its successive posi-

tions. The magnetic lines of force are thus found to be closed lines enclosing the wire and lying in planes perpendicular to the wire. One of these lines of force is shown in Fig. 3.16. The intensity of the magnetic field is proportional to that of the current in the wire. The direction of the magnetic force reverses when the direction of current flow in the wire reverses.

Consider then a battery B, Fig. 3.17, the terminals of which are connected by a wire A. The current i flowing in the closed circuit then produces a magnetic field around the latter, the magnetic lines of force of which are directed as shown by the arrows M in the figure. The closed circuit loop thus encloses a magnetic field flux, much in the same manner as the loop A in Fig. 3.15. If then the current intensity in the wire is varied, the magnetic field flux will vary accordingly. As in the case of Fig. 3.15, an electromotive force e will thereby be induced in the circuit loop, equal to the rate of change of the flux. This is expressed by the same equation (3.55) as above:

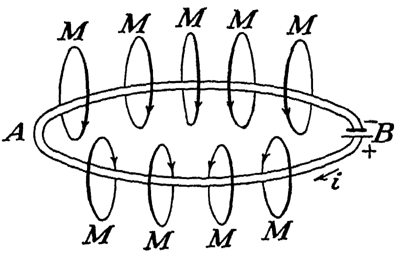

FIG. 3.17.—Magnetic field around a circuit loop.

$$e = \frac{d\phi}{dt} \tag{3.55}$$

and since the flux is proportional to the current intensity, this equation may be written

$$e = L\frac{di}{dt}. \tag{3.56}$$

The proportionality factor L is called the *inductance* of the circuit. The unit of inductance is the henry. A current varying at the rate of 1 amp. per sec. causes an electromotive force of 1 volt to be induced in a circuit having an inductance of 1 henry.

Reactance, Impedance.—The preceding equations (3.46, 3.54, and 3.56) show certain essential differences between the voltage-current relations in a resistance, a capacitance, and an inductance.

Referring first to the case of a simple resistance element, the equation

$$e = iR \tag{3.46}$$

relates the *magnitudes* of the voltage and current operating in the circuit, irrespective of their time dependence. In other words, the voltage and current are at all times in constant proportion to each other if the resist-

ance R is constant; the time functions representing them are thus of exactly the same form.

In the case of a condenser of constant capacitance C, the voltage e and current i are tied by the equation

$$i = C\frac{de}{dt} \tag{3.54}$$

whereby the magnitude of the current is proportional to the *time rate of change* of the voltage, but not to the voltage magnitude itself. Thus, for example, a constant voltage across the plates of a condenser does not produce any current in the circuit. Only when the condenser voltage varies with time does any current flow. In particular, a condenser voltage varying at a constant rate produces a current of constant intensity. In general, the ratio e/i of the instantaneous voltage and current values is therefore not constant, but varies with time. Consequently, the voltage and current are generally represented by time functions of different forms, and may have widely different wave shapes.

Similarly, for an inductance, the relation

$$e = L\frac{di}{dt} \tag{3.56}$$

shows that a constant current flowing through a constant inductance produces no voltage across the latter. Only when the current intensity varies, is there any voltage difference developed across the inductance. In particular, a current varying at constant rate produces a constant voltage across the coil. Here again, as in the case of a condenser, but contrary to that of a resistance, the ratio e/i of instantaneous voltage and current values generally varies with time. The voltage and current are then generally represented by time functions of different forms, and may have widely different wave shapes.

It follows that while the current and voltage in a circuit composed solely of resistance elements are related by an algebraic equation, they are tied by a differential equation whenever inductances or capacitances are included in the circuit. However, there is one instance in which, as in the case of a resistance, the current and voltage in an inductance or a capacitance are time functions of the same form, and in which therefore the voltage/current ratio has constant value. This is the case when either the current or the voltage is an exponential function of time. Both voltage and current are then functions of this same form, and their ratio is a constant.

Thus, consider a voltage

$$e = E_o\epsilon^{at}, \tag{3.57}$$

where ϵ is the base of Napierian logarithms, E_0 and a are constants, and t represents the time. When this voltage operates across an inductance L, Eq. (3.56) becomes

$$i = \frac{1}{L}\int e\,dt = \frac{E_0}{L}\int \epsilon^{at}\,dt = \frac{E_0}{aL}\epsilon^{at}, \tag{3.58}$$

which is of the same exponential form as the applied voltage Eq. (3.57). It follows that the voltage/current ratio is then equal to

$$\frac{e}{i} = \frac{E_0\epsilon^{at}}{(E_0/aL)\epsilon^{at}} = aL = \text{a constant}, \tag{3.59}$$

as in the case of a resistance.

Similarly, when the voltage Eq. (3.57) operates across a capacitance C, Eq. (3.54) becomes

$$i = C\frac{de}{dt} = CE_0\frac{d\epsilon^{at}}{dt} = aCE_0\epsilon^{at}, \tag{3.60}$$

which is of the same form as the applied voltage. The voltage/current ratio is then

$$\frac{e}{i} = \frac{E_0\epsilon^{at}}{aCE_0\epsilon^{at}} = \frac{1}{aC} = \text{a constant}, \tag{3.61}$$

as in the case of a resistance.

Now, if the exponential factor a is an imaginary number

$$a = j\omega \tag{3.62}$$

where j is equal to $\sqrt{-1}$ and ω is a real number, the exponential function, Eq. (3.57), can be written in trigonometric form (Euler's equation)

$$E_0\epsilon^{at} = E_0\epsilon^{j\omega t} = E_0(\cos\omega t + j\sin\omega t). \tag{3.63}$$

This expression is a sinusoidal time function of radian frequency ω or frequency $\omega/2\pi$ cycles per sec.

The factors aL and $1/aC$ of Eqs. (3.59) and (3.61) are then called the *reactances* of the inductance L and capacitance C, respectively. By introducing these reactance factors, the voltage/current relations in the three types of circuit elements are then expressed by similar algebraic equations

$$e = iR \tag{3.46}$$
$$e = iaL = iX_L \tag{3.64}$$
$$e = i\frac{1}{aC} = iX_C. \tag{3.65}$$

These reactances X_L and X_C are "imaginary" numbers. Relations involving them should, therefore, be handled in accordance with the rules of complex number algebra.

A circuit composed of a resistance R, inductance L, and capacitance C connected in series is then governed by the equation

$$\begin{aligned} e &= iR + iX_L + iX_C \\ &= i(R + X_L + X_C) = iZ, \end{aligned} \tag{3.66}$$

where the constant Z is called the *impedance* of the circuit. This impedance formally combines the resistance and reactance vector elements as though these were all simple resistance elements.

In a circuit composed of a resistance R, inductance L, and capacitance C connected in parallel, the impedance is calculated from the expression

$$\frac{1}{Z} = \frac{1}{R} + \frac{1}{X_L} + \frac{1}{X_C} \tag{3.67}$$

in the same manner as though the circuit were composed of parallel-connected resistance elements.

The calculation procedure for a network comprising any combination of resistances, inductances, and capacitances is then as follows.

1. Express the time dependence of the voltage in exponential form,

$$e = E_0 \epsilon^{j\omega t} \tag{3.68}$$

2. To every inductance L and capacitance C assign a reactance factor,

$$X_L = j\omega L \tag{3.69}$$

$$X_C = \frac{1}{j\omega C} \tag{3.70}$$

which are expressed in ohms when L and C are expressed, respectively, in henrys and farads and ω is expressed in radians per sec.

3. Treat these reactances like resistances, considering them as proportionality factors between voltage and current. Solve the problem, for example, by finding the currents in the various circuit branches or the voltages across the various circuit elements, as though the reactances were simple resistance elements, but using complex number algebra.

4. Applying Eq. (3.63), express the result in trigonometric form, and retain only the real terms of the solution, discarding the imaginary terms.[1]

[1] These imaginary terms have no physical significance, in the present treatment. They are necessarily added in Eq. (3.63) to convert a cosine function into exponential form, and must hence be deleted when the answer is converted back into trigonometric form. For a justification and explanation of this procedure, see, for instance, Vannevar Bush, "Operational Circuit Analysis," pp. 32–33, John Wiley & Sons, Inc., New York, 1937.

It should be noted that the reactance and impedance are functions of the frequency ω (or, more generally, of the time rate of change factor a) of the voltage and current functions. Since the inductive reactance $j\omega L$ and capacitive reactance $1/j\omega C$ vary in opposite directions when the frequency ω is changed, the impedance of the circuit may change with the frequency in a more or less complicated manner, depending on the arrangement of the circuit elements in the network. It follows that for an applied alternating voltage of constant amplitude the currents and voltages in the various branches of the network will vary with the frequency of the applied voltage. Use is made of this characteristic when it is desired to separate currents and voltages of different frequencies, or when effects of different magnitudes or phase are required, depending on the applied frequency.

It will be noted also from Eqs. (3.46), (3.54), and (3.56) that it is possible to obtain currents and voltages that are proportional either to the applied voltage or to its time derivative or time integral, depending on the circuit elements used and the arrangement of these elements. Application of these features will be found in later chapters, where so-called *differentiating* and *integrating filter networks* are discussed in relation to particular forms of servomechanisms.

CHAPTER IV

ANALYSIS OF SERVOMECHANISMS WITH VISCOUS OUTPUT DAMPING

The preceding chapters contain a qualitative description of the simpler types of servomechanisms and control systems and of certain devices commonly used with these. However, actual design and practical applications require a more complete and accurate knowledge of the operating characteristics of these systems, obtainable only through quantitative analysis of their properties in mathematical terms.[1]

The purpose of the systems discussed is to drive an output load in such manner that its position at all times corresponds to that of the input member of the system. One of the problems encountered is to determine the error between the instantaneous positions of the input and output members, under given varying conditions of input speed. Another problem relates to the design of a system, that is to say, the determination and proportioning of its several parameters, to meet specified operating requirements.

The present chapter is devoted to the analysis of one of the simplest types of servomechanisms, such as was described previously in a general way. More advanced types, in which the damping of transient errors can be increased without introducing any additional steady-state error, are considered in the chapters that follow.

Servomechanism with Viscous Output Damping.—The system to be studied here is represented schematically in Fig. 4.1. It comprises an input member, an output member connected to the load, a differential device to indicate the difference or error between the positions of the input and output members, and a controller which drives the load by producing a torque that is directly proportional to the error. The output friction includes friction in the servomotor embodied in the controller, friction in the load, and in the couplings and gears (not shown in Fig. 4.1) between the motor and load. This friction is symbolized in the diagram as a mechanical damper, but may assume other forms, as dis-

[1] The development of the concept of the transient analysis of servomechanisms is attributable to the pioneer work of Dr. H. L. Hazen of the Massachusetts Institute of Technology (see *J. Franklin Inst.*, September and November, 1934). This work was continued by Dr. Gordon S. Brown also of the Massachusetts Institute of Technology, and by H. K. Weiss, C. S. Draper, and others. Principles of the transient-analysis method are set forth in this and the four following chapters.

cussed later. The friction force is assumed to be directly proportional to the output speed, as is the case for purely viscous damping.

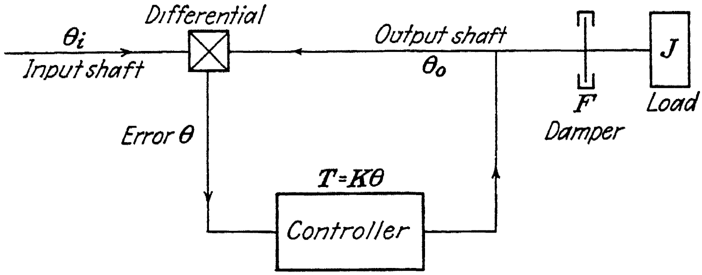

FIG. 4.1.—Simple servomechanism with viscous output damping.

In the analysis given below the following symbols are used to represent the quantities associated with the servo system:

Symbol	Quantity	C.g.s. units	F.p.s. units
θ_i ...	Input angle	Radians	Radians
θ_o.	Output angle	Radians	Radians
θ.. .. .	Error angle	Radians	Radians
K......	Output torque per unit error angle	$\frac{\text{Dyne-cm.}}{\text{Radian}}$	$\frac{\text{Ft.-lb.}}{\text{Radian}}$
F	Friction torque per unit output speed	$\frac{\text{Dyne-cm.}}{\text{Radians per sec.}}$	$\frac{\text{Ft.-lb.}}{\text{Radians per sec.}}$
J	Output moment of inertia	Gm.-cm.^{2}	Slug-ft.^{2}
t	Time	Seconds	Seconds

Equation of the Problem.—The operating conditions of the system described above are expressed by stating that the sum of the forces acting on the system is equal to zero. In other words, the accelerating forces must equal the retarding forces. In the present instance, the accelerating force is the torque $K\theta$ produced by the controller. This must equal the sum of the retarding inertia force (which is the product $J\, d^2\theta_o/dt^2$ of the output moment of inertia and the output angular acceleration) and the retarding viscous friction drag (product $F\, d\theta_o/dt$ of the friction coefficient and the output speed). Thus

$$K\theta = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt}. \qquad (4.1)$$

Since the error θ is the difference between the input and output angular positions θ_i and θ_o

$$\theta = \theta_i - \theta_o, \qquad (4.2)$$

Eq. (4.1) may be rewritten in terms of the input and error angles only, by substituting Eq. (4.2) in Eq. (4.1).

$$J\frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt} + K\theta = J\frac{d^2\theta_i}{dt^2} + F\frac{d\theta_i}{dt}. \tag{4.3}$$

Step Input Function.—In order to compare the transient responses of various systems, it is desirable to choose an input function $\theta_i(t)$ of known characteristics, and apply it to all such systems. In the investigation of this and other servomechanisms, an input function will therefore be considered in which the input member is motionless until some instant when it starts to move continuously at constant angular velocity ω_1, the input angle θ_i thereby increasing linearly with time. Time is measured

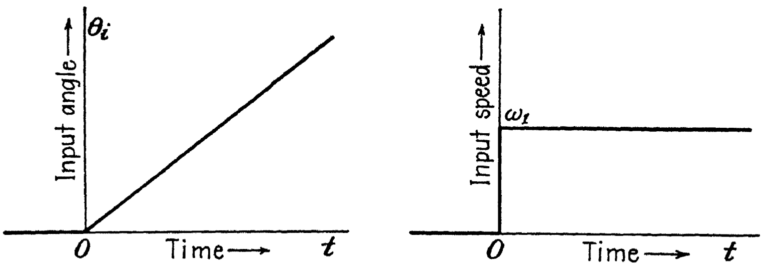

FIG. 4.2.—Graphs of input angle and input speed for step velocity input.

from the instant of the beginning of motion; this condition is represented graphically in Fig. 4.2, and may be expressed by the equations

$$\begin{cases} \theta_i = 0, & t < 0 \qquad (4.4) \\ \theta_i = \omega_1 t, & t \geq 0. \qquad (4.5) \end{cases}$$

If the entire system is at rest before the input member is set in motion, the error θ may be assumed to be zero up to the starting instant. At the instant $t = 0$ where the input speed jumps from zero to the constant value ω_1, a transient is initiated. If the system is stable, this transient will die out, leaving only a steady-state error.

For an input function of the type described, the behavior of the system after the input member has been set in motion (*i.e.*, for $t > 0$) is expressed by a differential equation obtained by substituting the first and second time derivatives of the function Eq. (4.5) in the right-hand member of Eq. (4.3). These derivatives are, respectively,

$$\frac{d\theta_i}{dt} = \omega_1, \tag{4.6}$$

$$\frac{d^2\theta_i}{dt^2} = 0. \tag{4.7}$$

Placing these values in Eq. (4.3), the equation becomes

$$J\frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt} + K\theta = F\omega_1. \tag{4.8}$$

The problem of finding the error θ from this equation will be treated in several steps. First, the steady-state error will be determined. Next, the general transient solution will be found. This solution will contain arbitrary constants, which are to be determined later. Finally, the steady-state and transient solutions will be combined, the complete solution being the sum of the two partial solutions previously found. The arbitrary factors that appear in the solution can then be determined from the known conditions of the system at the time origin ($t = 0$), when the input member starts to move.

Steady-state Error.—The steady-state error of a servo system can be found for any specified input function by a rather complex mathematical process.[1] However, it may be found quite easily when the input function is simple, as is the case for the step velocity input function considered here.

While the steady-state error may be taken as any time function that will satisfy Eq. (4.8), the simplest solution is that of a constant error. This solution will obtain if it is assumed that the output speed is equal to the input speed after a sufficiently long time has elapsed for the transient to die out and for the system to reach a stable operating condition; while the input member is being driven at constant speed from the very instant of starting, a certain time interval is required for the output member to accelerate and settle to this same speed. The position of the output member finally lags behind that of the input member by a certain amount, the *steady-state error*.

The steady-state error being constant, its first and second time derivatives (or rates of change) are zero. Equation (4.8) then reduces to

$$K\theta_S = F\omega_1 \qquad \text{or} \qquad \theta_S = \frac{F\omega_1}{K}, \tag{4.9}$$

which expresses the steady-state error θ_S as a function of the constant input speed, friction coefficient, and controller constant. Note that this steady-state error corresponds only to the case of constant input velocity.

Transient Error—The generalized expression of the transient error is determined by the characteristics of the servomechanism itself and is

[1] See SOKOLNIKOFF, I. S., and E. S. SOKOLNIKOFF, "Higher Mathematics for Engineers and Physicists," 2d ed., pp. 295–299, McGraw-Hill Book Company, Inc., New York, 1941.

independent of the form of the input function. Consequently, a valid solution can be obtained with the input function in Eq. (4.3) set at zero.

$$J\frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt} + K\theta = 0. \tag{4.10}$$

This equation is of exactly the same form as Eq. (3.43) encountered in Chap. III. It was stated that the solution for such equations is an exponential function of time, so that the transient error θ_T is, here also, expressed in the form

$$\theta_T = A\epsilon^{pt} \tag{4.11}$$

where ϵ is the base of Napierian logarithms and the factors p and A are constants yet to be determined.

To find the value of p, Eq. (4.11) and its first and second time derivatives are substituted in Eq. (4.10). These derivatives are, respectively,

$$\frac{d\theta_T}{dt} = pA\epsilon^{pt}, \tag{4.12}$$

$$\frac{d^2\theta_T}{dt^2} = p^2A\epsilon^{pt}. \tag{4.13}$$

Equation (4.10) then becomes

$$Jp^2A\epsilon^{pt} + FpA\epsilon^{pt} + KA\epsilon^{pt} = 0 \tag{4.14}$$

or

$$A\epsilon^{pt}(Jp^2 + Fp + K) = 0. \tag{4.14a}$$

It is seen that, irrespective of the value of the factor A, this equation is satisfied when the quantity in brackets is equal to zero.

$$Jp^2 + Fp + K = 0. \tag{4.15}$$

In other words, Eq. (4.14) is satisfied when p has one of the values determined by the quadratic equation (4.15). Solving this for p, the values are obtained

$$p_1 = -\frac{F}{2J} + \sqrt{\frac{F^2}{4J^2} - \frac{K}{J}}, \tag{4.16}$$

$$p_2 = -\frac{F}{2J} - \sqrt{\frac{F^2}{4J^2} - \frac{K}{J}}. \tag{4.17}$$

Writing these two values of p into Eq. (4.11), two solutions for θ are obtained,

$$\theta_{T_1} = A_1\epsilon^{\left(-\frac{F}{2J} + \sqrt{\frac{F^2}{4J^2} - \frac{K}{J}}\right)t}, \tag{4.18}$$

$$\theta_{T_2} = A_2\epsilon^{\left(-\frac{F}{2J} - \sqrt{\frac{F^2}{4J^2} - \frac{K}{J}}\right)t}. \tag{4.19}$$

The general transient solution is the sum of these two values.

$$\theta_T = A_1\epsilon^{\left(-\frac{F}{2J}+\sqrt{\frac{F^2}{4J^2}-\frac{K}{J}}\right)t} + A_2\epsilon^{\left(-\frac{F}{2J}-\sqrt{\frac{F^2}{4J^2}-\frac{K}{J}}\right)t}. \tag{4.20}$$

Depending on the relative values of the factors F, J, and K, the quantity under the radical may be positive, negative, or zero. As will be shown more fully later, these three cases correspond, respectively, to an overdamped, underdamped (oscillatory), and critically damped system. The overdamped case will not be studied here, since it involves such a slow response of the system to input speed variations as to be of no practical use in servo applications. High-speed response is one of the desirable features of a servomechanism and therefore the quantity under the radical must be either zero or negative.

For simpler writing, let then

$$a \equiv \frac{F}{2J} \tag{4.21}$$

$$jb \equiv \sqrt{\frac{F^2}{4J^2} - \frac{K}{J}} \equiv j\sqrt{\frac{K}{J} - \frac{F^2}{4J^2}}, \tag{4.22}$$

where $j \equiv \sqrt{-1}$. Equation (4.20) thus becomes

$$\theta_T = A_1\epsilon^{(-a+jb)t} + A_2\epsilon^{(-a-jb)t}, \tag{4.23}$$

or

$$\theta_T = A_1\epsilon^{-at}\epsilon^{jbt} + A_2\epsilon^{-at}\epsilon^{-jbt}, \tag{4.24}$$

which may be written

$$\theta_T = \epsilon^{-at}(A_1\epsilon^{jbt} + A_2\epsilon^{-jbt}). \tag{4.25}$$

Substituting the well-known trigonometric identities,

$$\epsilon^{jbt} = \cos bt + j \sin bt \tag{4.26}$$

$$\epsilon^{-jbt} = \cos bt - j \sin bt, \tag{4.27}$$

Eq. (4.25) may be expressed as

$$\theta_T = \epsilon^{-at}[A_1(\cos bt + j \sin bt) + A_2(\cos bt - j \sin bt)], \tag{4.28}$$

or

$$\theta_T = \epsilon^{-at}[(A_1 + A_2) \cos bt + j(A_1 - A_2) \sin bt]. \tag{4.29}$$

Defining

$$B_1 = A_1 + A_2 \tag{4.30}$$

$$B_2 = j(A_1 - A_2), \tag{4.31}$$

Eq. (4.25) becomes

$$\theta_T = \epsilon^{-at}(B_1 \cos bt + B_2 \sin bt). \tag{4.32}$$

Determination of the two arbitrary constants B_1 and B_2 will be accomplished in the following paragraph, in order to obtain the complete solution of Eq. (4.3).

Complete Solution of the Equation.—A complete solution of Eq. (4.3), if the angular position θ_i of the input member varies as a function of time in the manner defined by the relations, Eqs. (4.4) and (4.5), is obtained by adding the steady-state and transient solutions, Eqs. (4.9) and (4.32), and by evaluating the constants B_1 and B_2 from the conditions of the servo system at the starting instant.

The complete expression of the error θ is then

$$\theta = \theta_S + \theta_T = \frac{F\omega_1}{K} + \epsilon^{-at}\,(B_1 \cos bt + B_2 \sin bt). \tag{4.33}$$

In order to find the two constants B_1 and B_2, two equations are necessary. These equations are obtained by expressing the known conditions of the servo system at the time origin $t = 0$, when the input member of the system is suddenly started. Since, prior to this starting instant, the entire system is at rest, the input and output member displacements θ_i, θ_o, as well as the error are zero.

$$\theta = \theta_i = \theta_o = 0, \qquad t \leq 0 \tag{4.34}$$

Also, since only finite forces can be applied to the output member, it is seen that not only the output displacement θ_o but also the output speed (first time derivative of the displacement) are zero at the starting instant of the input member.

$$\frac{d\theta_o}{dt} = 0, \qquad t = 0. \tag{4.35}$$

In order to apply this condition to Eq. (4.33), its equivalent in terms of the error θ must be found. Recalling Eqs. (4.2) and (4.5),

$$\theta = \theta_i - \theta_o, \tag{4.2}$$

$$\theta_i = \omega_1 t, \qquad t \geq 0, \tag{4.5}$$

and differentiating, the relations are obtained

$$\frac{d\theta}{dt} = \frac{d\theta_i}{dt} - \frac{d\theta_o}{dt}, \tag{4.36}$$

$$\frac{d\theta_i}{dt} = \omega_1, \qquad t \geq 0. \tag{4.37}$$

Substituting Eqs. (4.35) and (4.37) in Eq. (4.36),

$$\frac{d\theta}{dt} = \omega_1, \qquad t = 0. \tag{4.38}$$

Eqs. (4.34) and (4.38) are sufficient for determining the constants B_1 and B_2 of Eq. (4.33).

Applying condition (4.34) to Eq. (4.33),

$$\theta_{t=0} = 0 = \frac{F\omega_1}{K} + B_1 \tag{4.39}$$

from which

$$B_1 = \frac{-F\omega_1}{K}. \tag{4.40}$$

For applying condition (4.38), it is first necessary to differentiate Eq. (4.33).

$$\frac{d\theta}{dt} = \epsilon^{-at}(-B_1 b \sin bt + B_2 b \cos bt - B_1 a \cos bt - B_2 a \sin bt). \tag{4.41}$$

At the time origin ($t = 0$), this reduces to

$$\frac{d\theta}{dt} = B_2 b - B_1 a, \qquad t = 0, \tag{4.42}$$

or, in view of Eq. (4.38),

$$\omega_1 = B_2 b - B_1 a. \tag{4.43}$$

Substituting for B_1 the value obtained in Eq. (4.40), Eq. (4.43) becomes

$$\omega_1 = B_2 b + \frac{Fa\omega_1}{K}, \tag{4.44}$$

or

$$B_2 = \frac{1}{b}\left(\omega_1 - \frac{Fa\omega_1}{K}\right) \tag{4.45}$$

The values of B_1 and B_2 found in Eqs. (4.40) and (4.45) may now be written in Eq. (4.33).

$$\theta = \frac{F\omega_1}{K} + \epsilon^{-at}\left[\frac{-F\omega_1}{K}\cos bt + \frac{1}{b}\left(\omega_1 - \frac{Fa\omega_1}{K}\right)\sin bt\right]. \tag{4.46}$$

This expression is the complete solution of the problem. It gives the error angle as a function of the input speed and the system parameters, the constants a and b being as defined in Eqs. (4.21) and (4.22). Depending on the values of the parameters, several cases may arise, the more important of which will be discussed. These are, respectively,

1. The undamped case, where the system is considered to be frictionless. The error then has an average value of zero, but oscillates continuously with an amplitude that is proportional to the input speed.

2. The critically damped case where the steady-state error has a sizable value but is reached without oscillation.

3. An intermediate case (underdamped) where the steady-state error, lthough finite, is less than in the critically damped case. However, he steady-state error value is then reached only after a more or less xtended transient period, during which the error and the output speed scillate about their ultimate (or steady-state) values.

4. The overdamped case, mentioned here for completeness only, ince it is not generally encountered in practical servo systems.

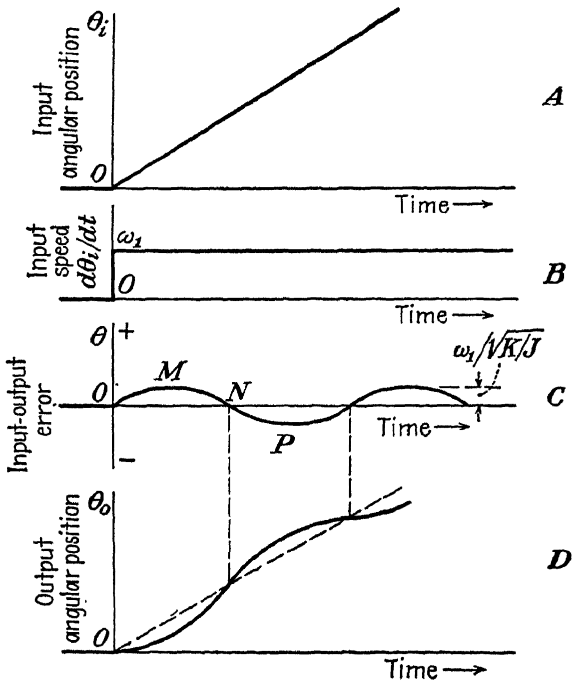

FIG. 4.3.—Operation of undamped servo system.

Undamped System.—Suppose that the damping coefficient F of the ervo system of Fig. 4.1 is equal to zero. Equations (4.21) and (4.22) hen become

$$a = 0, \tag{4.47}$$

$$b = \sqrt{\frac{K}{J}}, \tag{4.48}$$

ınd Eq. (4.46) reduces to

$$\theta = \frac{\omega_1}{\sqrt{\frac{K}{J}}} \sin \sqrt{\frac{K}{J}}\, t. \tag{4.49}$$

Expressed in words, with the step input function considered here, the error is a constant amplitude, sinusoidal function of time. This is

illustrated in Fig. 4.3; at the time origin ($t = 0$), the input member of the system is suddenly set in motion at constant angular speed ω_1 (diagram B), so that the input angular position θ_i increases linearly with time (diagram A). The output member being originally at rest, this produces an *error* between the positions of the input and output members, which at first increases with time (curve portion OM, diagram C). This error causes the controller (Fig. 4.1) to develop a proportional torque, which sets the output member in motion. The output member accelerates, at a rate that is proportional to the torque, and inversely proportional to the inertia, until the output member angular position θ_o, which had lagged behind that of the input member, overtakes the latter and reduces the error to zero (point N).

Although this reduces to zero the torque developed by the controller, the output inertia (load, motor, gears, etc.) of the system causes the output member to *overshoot* this position. The output member then advances ahead of the input member, producing a negative error (point P); and the controller therefore develops a negative or retarding torque. This tends to decelerate the output member and again reduce the input-output error to zero. The output member, once more, overshoots the coincidence position, and the oscillation process, also called *hunting*, repeats itself indefinitely.

As shown by Eq. (4.49), the *amplitude* of oscillation is equal to $\omega_1/\sqrt{K/J}$. The *frequency* of oscillation, expressed in radians per second, is equal to

$$\omega_n = \sqrt{\frac{K}{J}} \tag{4.50}$$

and is called the *natural frequency* of the servomechanism.

Critically Damped System.—Equation (4.46) may be rewritten

$$\theta = \frac{F\omega_1}{K} - \epsilon^{-at}\left[\frac{F\omega_1}{K}\cos bt + \left(\frac{Fa\omega_1}{K} - \omega_1\right)\frac{\sin bt}{b}\right]. \tag{4.51}$$

Also, according to Eq. (4.22), the value of b may be made to approach zero by letting $F^2/4J^2$ tend toward K/J

$$\frac{F^2}{4J^2} \to \frac{K}{J}. \tag{4.52}$$

As the factor b tends toward zero, the terms $\cos bt$ and $(\sin bt)/b$ tend,[1] respectively, toward 1 and t.

$$b \to 0 \begin{cases} \cos bt \to 1 & (4.53) \\ \dfrac{\sin bt}{b} \to t. & (4.54) \end{cases}$$

[1] As b tends toward zero, $\sin bt$ is virtually equal to bt. The expression $(\sin bt)/b$ may then be written as bt/b, or t, as stated in Eq. (4.54).

Substituting these values in Eq. (4.51),

$$\theta = \frac{F\omega_1}{K} - \epsilon^{-at}\left[\frac{F\omega_1}{K} + \left(\frac{Fa\omega_1}{K} - \omega_1\right)t\right]. \tag{4.55}$$

From this expression it is seen that in this limiting case the error approaches a constant steady-state value without oscillations. This condition, illustrated in Fig. 4.4, is known as *critical damping*. From Eq. (4.52), it follows that the value of output damping F_c for which the system will be critically damped is expressed

$$F_c = 2\sqrt{KJ}. \tag{4.56}$$

Underdamped System.—For practical applications the undamped and critically damped cases studied above represent limit cases. An undamped system has the advantage of having an *average* error equal to zero, but offers the disadvantage of steady oscillation or hunting. On the other hand, a critically damped system, while free from oscillation, has the drawback of causing a larger steady-state error to exist between the input and output than can be tolerated in applications where great accuracy is required. A compromise between these conditions is therefore generally desirable, where a tolerably small error is obtained at the cost of a temporary or transient oscillation occurring whenever the input speed is suddenly changed. This compromise is represented by the *underdamped* system in which the damping, although not inexistent or zero, is less than that corresponding to critical damping.

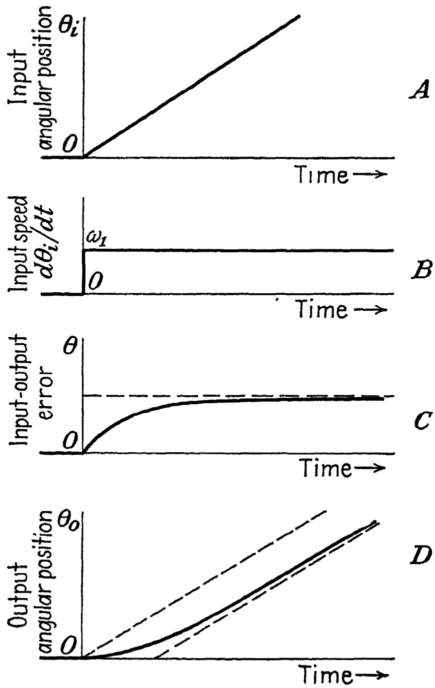

FIG. 4.4.—Operation of critically damped servo system.

Since the behavior of such systems depends essentially on the *relative* values of the characteristic factors of the system, a new parameter c is introduced. This parameter is equal to the ratio of the actual damping factor F of the system to the damping factor F_c that would be required to critically damp the system, as defined above. Thus

$$c \equiv \frac{F}{F_c}. \tag{4.57}$$

Combining Eqs. (4.56) and (4.57) gives

$$F = 2c\sqrt{KJ} \tag{4.58}$$

while the other parameters of the system, as defined in Eqs. (4.21), (4.22), and (4.50), may then be written

$$a = c\omega_n \tag{4.59}$$
$$b = \omega_n\sqrt{1-c^2} \tag{4.60}$$
$$\frac{F}{K} = \frac{2c}{\omega_n}. \tag{4.61}$$

Dimensionless Form of Equations.[1]—In order to obtain equations better suited for practical applications, the expressions found in the preceding paragraphs may be inserted in the equations established previously. Inserting Eqs. (4.59) and (4.61) in Eq. (4.55) for the *critically damped case*, where $c = 1$,

$$\theta = \frac{2\omega_1}{\omega_n} - \epsilon^{-\omega_n t}\left[\frac{2\omega_1}{\omega_n} + (2\omega_1 - \omega_1)t\right], \tag{4.62}$$

or

$$\theta\frac{\omega_n}{\omega_1} = 2 - (2 + \omega_n t)\epsilon^{-\omega_n t}. \tag{4.63}$$

Similarly, Eqs. (4.59), (4.60), and (4.61) may be written in Eq. (4.46) for the *general oscillatory case*.

$$\theta = \frac{2c\omega_1}{\omega_n} - \epsilon^{-c\omega_n t}\left[\frac{2c\omega_1}{\omega_n}\cos\omega_n\sqrt{1-c^2}\,t + \left(\frac{2c^2\omega_1}{\omega_n\sqrt{1-c^2}} - \frac{\omega_1}{\omega_n\sqrt{1-c^2}}\right)\sin\omega_n\sqrt{1-c^2}\,t\right] \tag{4.64}$$

or

$$\theta\frac{\omega_n}{\omega_1} = 2c - \left[2c\cos\omega_n\sqrt{1-c^2}\,t + \frac{2c^2-1}{\sqrt{1-c^2}}\sin\omega_n\sqrt{1-c^2}\,t\right]\epsilon^{-c\omega_n t}. \tag{4.65}$$

Equations (4.63) and (4.65) express in dimensionless form the error of a viscous-damped servomechanism in which the input member is suddenly set in motion at constant velocity. The right-hand member of each of these equations contains

[1] The purpose of using dimensionless parameters is explained in a later paragraph. Dimensionless parameters similar to those employed here are used by S. W. Herwald, Considerations in Servomechanism Design, *Trans. Am. Inst. of Elec. Eng.*, vol. 63, December, 1944. Dimensionless servo equations were first given by H. L. Hazen, Theory of Servomechanisms, *J. Franklin Inst.*, September, 1934.

1. A constant term, which represents the steady-state error.
2. An exponential term, which in view of its negative exponent decreases in magnitude when the time t increases. This term represents the transient error, and becomes negligibly small for large values of t. Only the constant, steady-state term then remains in the equation.

It should be noted that Eq. (4.65) applies for values of the damping ratio c between zero and unity. This equation may be written in the form

$$\theta \frac{\omega_n}{\omega_1} = 2c - \left[2c \cosh \omega_n \sqrt{c^2 - 1}\, t + \frac{2c^2 - 1}{\sqrt{c^2 - 1}} \sinh \omega_n \sqrt{c^2 - 1}\, t\right] \epsilon^{-c\omega_n t} \quad (4.65a)$$

applicable to the case where $c > 1$, as occurs when the servo system is overdamped.

Application of these dimensionless equations is described in a later paragraph.

Problem.—In order to familiarize himself with the various quantities involved in the design and calculation of servomechanisms, the reader may work out the following problem.

A servomechanism whose moment of inertia is 10×10^{-6} slug-ft.[2] has a retarding output friction coefficient of 400×10^{-6} ft.-lb per radian per sec. output velocity, and employs a controller that produces a torque of 0.004 ft.-lb. per radian error. Plot the error angle of the servo as a function of time when the input member is suddenly started at a constant angular velocity of 10 r p.m.

Solution: (The reader should treat the problem himself, before reading its solution given below.)

Using the symbols employed in the text,

$J = 10 \times 10^{-6}$ slug-ft.^{2}

$F = 400 \times 10^{-6}$ ft.-lb. per radian per sec.

$K = 4 \times 10^{-3} = 4{,}000 \times 10^{-6}$ ft.-lb. per radian

$\omega_1 = 10$ r.p.m. $= \dfrac{10 \times 2\pi}{60}$ radians per sec. (or approximately 1 radian per sec.).

From Eq. (4.50)

$$\omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{4{,}000 \times 10^{-6}}{10 \times 10^{-6}}} = \sqrt{400} = 20 \text{ radians per sec.}$$
$$= \text{about 3 cycles per sec.}^1$$

From Eq. (4.56)

$$F_c = 2\sqrt{KJ} = 2\sqrt{4{,}000 \times 10^{-6} \times 10 \times 10^{-6}} = 2\sqrt{4 \times 10^{-8}}$$
$$= 400 \times 10^{-6} \text{ ft.-lb. per radian per sec.}$$

[1] One cycle corresponds to 360 deg., or 2π radians. Therefore 1 radian is equivalent to $360/2\pi = 57.3$ deg., or 0.159 cycle. Thus, in the present problem, a frequency of 20 radians per sec. can also be expressed as equal to $20/2\pi = 3$ cycles per sec., approximately, as stated in the text.

From Eq. (4.57)

$$c = \frac{F}{F_c} = \frac{400 \times 10^{-6}}{400 \times 10^{-6}} = 1.$$

The system being critically damped, since the damping ratio just found is equal to unity, Eq. (4.63) is used for determining the error.

$$\theta \frac{\omega_n}{\omega_1} = 2 - (2 + \omega_n t)\epsilon^{-\omega_n t} \tag{4.63}$$

or

$$\begin{aligned}\theta &= \frac{\omega_1}{\omega_n}[2 - (2 + \omega_n t)\epsilon^{-\omega_n t}] \\ &= \tfrac{1}{20}[2 - (2 + 20t)\epsilon^{-20t}] \\ &= \tfrac{1}{10}[1 - (1 + 10t)\epsilon^{-20t}] \text{ radians.}\end{aligned}$$

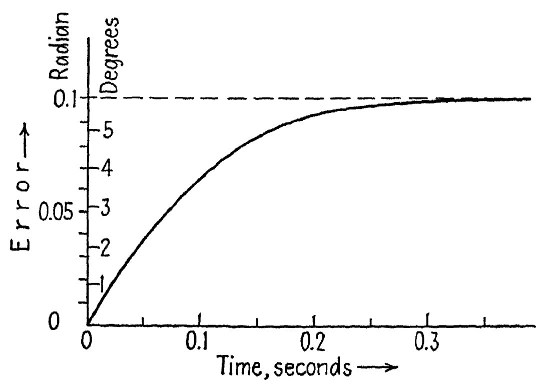

FIG. 4.5.—Error-time curve for critically damped servo discussed in problem

Calculating this expression for successively increasing values of time, the corresponding instantaneous values of the error θ expressed in radians are obtained. These are listed below, together with their equivalents in degrees.

Time t, sec.	Error θ, radians	Error θ deg.
0	0	0
0.05	0.041	2.35
0.10	0.073	4.22
0.15	0.087	4.98
0.20	0.094	5.41
0.25	0.097	5.57
0.30	0.099	5.64
∞	0.1	5.73

These values are plotted in the diagram of Fig. 4.5. The curve represents, as a function of time, the angular error between the positions of the input and output members when the input member is *suddenly* started at a speed of 10 r.p.m. The output member reaches this same speed only some time later, after which it maintains a con-

stant relative position with respect to the input member. The error then remains constant. It will be noted that 0.3 sec. after the starting instant, the error has reached 99 per cent of its steady-state value of 5.73 deg.

Discussion and Use of Dimensionless Equations.—The curve of Fig. 4.5 shows the input-output error in degrees or in radians as a function of the time expressed in seconds and for the particular conditions and system considered. In other words, the actual error and time values are shown in the graph for a servo system that has a natural frequency $\omega_n = 20$ radians per sec. (equivalent to about 3 cycles per sec.) and a damping ratio $c = 1$, driven at a suddenly applied input speed $\omega_1 = 10$ r.p.m.

It may be seen from Eq. (4.63) that if the input speed ω_1 had been 20 r.p.m. instead of 10 r.p.m. (*i.e.*, twice the value used in the preceding example), the error would at every instant be twice as large as that shown in Fig. 4.5. The curve obtained would then be similar to that of the figure but with its ordinates twice as high. Indeed, the original curve could be used to represent the new operating conditions, by simply changing the scale along the vertical *error* axis. Another procedure, which makes it possible to use the same curve for *any* value of the input speed ω_1, is to plot the values of θ/ω_1 along the vertical axis instead of the values of the error θ itself.

On the other hand, if a different servo system had been considered, one having a natural frequency ω_n equal to, say, 10 radians per sec. instead of 20 radians per sec., the error θ would according to Eq. (4.63) be equal to twice the error found previously. Nevertheless, the same error plot will be obtained as in the original case, by plotting the values of $\theta(\omega_n/\omega_1)$ along the vertical axis. However, if $\theta(\omega_n/\omega_1)$ is plotted against *actual* values of the time t, the curve corresponding to $\omega_n = 10$ will stretch out horizontally twice as far as the curve corresponding to $\omega_n = 20$. The two curves can be made to coincide with each other by plotting them against values of the product $\omega_n t$ instead of against actual values of t.

Thus, by plotting $\theta(\omega_n/\omega_1)$ against $\omega_n t$ instead of plotting the actual values of the error θ against actual values of the time t, the curve of Fig. 4.5 can be made a universal curve valid for any viscous-damped servo system with a damping ratio $c = 1$. Such a curve is shown in Fig. 4.6.

The same principle applies to systems having other damping ratios, there being one universal curve for every value of this ratio. Such curves are shown in Fig. 4.6, as calculated from Eq. (4.65), for values of c equal to 0.5, 0.25, and 0.1, respectively. Like the curve that corresponds to a damping ratio $c = 1$, these curves do not show directly the relation between the error θ and the time t, but the relation between the error multiplied by the ratio ω_n/ω_1 and the time multiplied by ω_n.

It should be noted that when using either of the two equations (4.63) and (4.65), or the curves of Fig. 4.6 that represent these equations, both expressions $\theta(\omega_n/\omega_1)$ and $\omega_n t$ must be dimensionless. For making $\theta(\omega_n/\omega_1)$ dimensionless, the error θ may be expressed in any convenient angular measure (degrees, radians, revolutions, etc.). The input speed ω_1 is expressed in units of angle per unit of time: any unit of time may be used

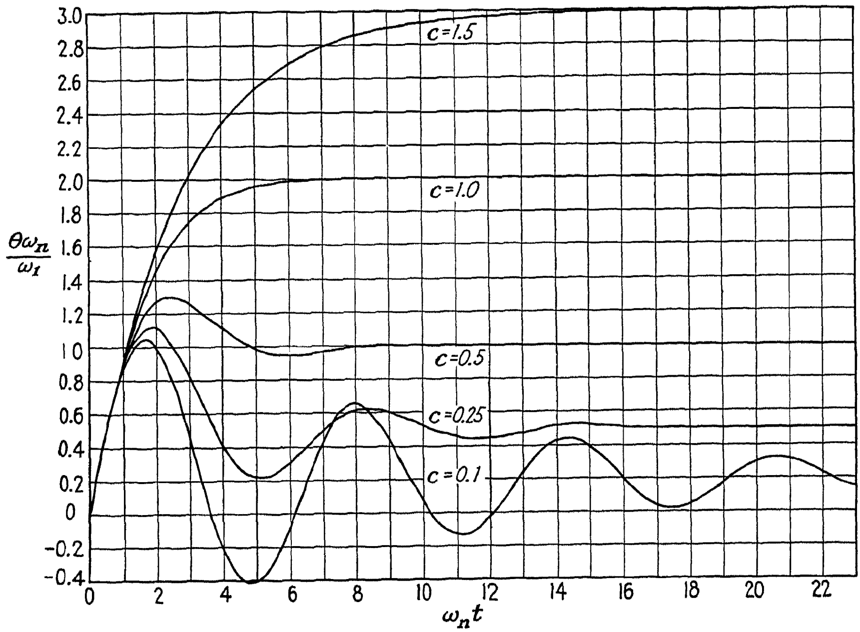

FIG. 4.6.—Dimensionless error-time curves for viscous-damped servomechanism subjected to a step input velocity function.

(seconds, minutes, etc.), but the unit of angle[1] must be the same as that used for the error θ. Finally, the natural frequency ω_n must be expressed in radians per unit of time, the unit of time being the same as that used for expressing the input speed ω_1. For making $\omega_n t$ dimensionless, ω_n must be expressed in radians per unit of time and t must be measured in this same unit of time.[2]

[1] These remarks apply to rotary systems. If the input and output members of the system are subject to translatory motion, the unit of length entering in the measure of θ and ω_1 (position error and input speed) must be the same for both quantities.

[2] The expression $\theta(\omega_n/\omega_1)$ is then dimensionless because the angle dimensions of θ and ω_1 and the time dimensions of ω_n and ω_1 cancel out, while the radian measure of ω_n is itself a dimensionless ratio of arc length to radius length.

The expression $\omega_n t$ is dimensionless because the time dimensions of ω_n and t cancel out, and, as just recalled, the radian measure of ω_n is a dimensionless ratio.

The curves of Fig. 4.6 illustrate the fact, mentioned before, that the steady-state error decreases with the damping ratio *c*.

At the same time, a transient oscillation is originated at the starting instant, the maximum value (overshoot) and duration of which increase as the damping ratio is made smaller. Thus, Fig. 4.7 shows the value of the maximum error overswing θ_{max}, compared to the steady-state error

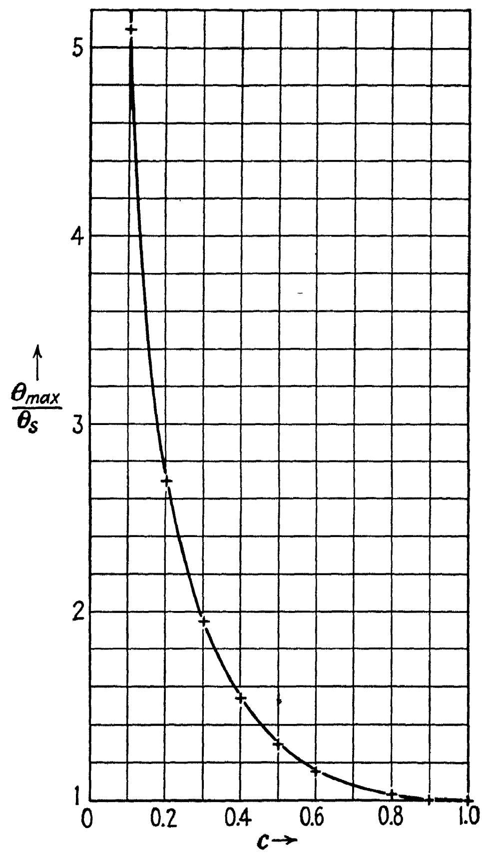

FIG. 4.7.—Ratio of maximum transient error to steady-state error as a function of damping ratio in a viscous-damped servo with a step input velocity.

value θ_S as a function of the damping ratio *c*. The two curves of Fig. 4.8 show, as a function of the damping ratio *c*, the amount of time $\omega_n t$ required for the transient oscillation amplitude to decay, respectively, to $1/\epsilon$ (or 0.368) and $1/10$ of its maximum value.

There are several methods by which the damping ratio may be changed. One of these consists in changing the friction damping *F* of the system, which is easily done when such damping is obtained by means of an eddy-current damper. Another method involves a change of the

friction value F_c required to damp the system critically. Thus, rewriting relation, Eq. (4.58), in the form

$$c = \frac{F}{2\sqrt{KJ}}, \tag{4.58a}$$

it is seen that c can be determined by a proper choice of the controller factor K that expresses the value of output torque produced by the controller for a unit error angle, as indicated by the differential device.

Redrawing in Fig. 4.9 the servo system shown in Fig. 4.1, the differential device may be one of the electromechanical synchro repeaters

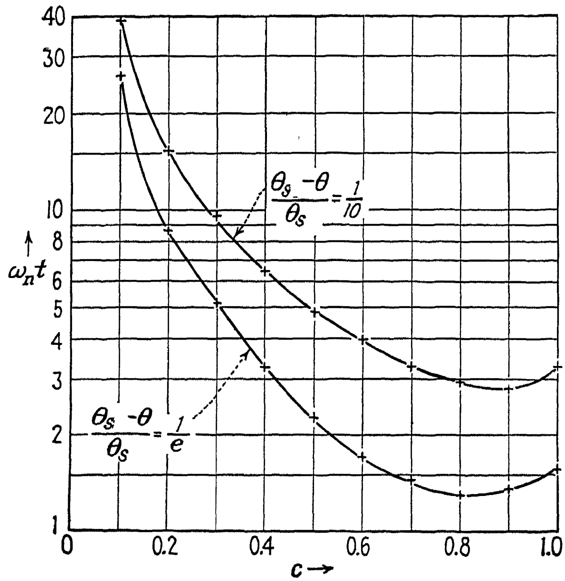

FIG. 4.8.—Transient decay time of viscous damped servo as a function of damping ratio.

described in Chap. II. This feeds into the controller an error voltage that is proportional to the mechanical input-output error angle. The controller may consist of an electrical amplifier through which the error voltage, suitably amplified, is applied to a servo motor. If this motor develops a torque directly proportional to the voltage applied to it, the controller factor K is seen to be directly proportional to the voltage gain of the amplifier. It is then an easy matter to obtain any desired value for the factor K.

The process, however, of reducing the steady-state error by reducing the damping ratio c cannot be extended indefinitely. The reason is that, as shown by the curves of Fig. 4.6, the transient oscillation then increases in amplitude and duration and indeed never dies down when c is made equal to zero.

Another point that should be well understood is that a steady-state error, damping, and friction go hand in hand in the type of servo considered here. Since some damping is necessary, in practice, to reduce the transient oscillation, a corresponding amount of friction must be provided. This, in turn, was shown to produce a steady-state error between the relative positions of the input and output member positions. If there were no such error, the controller would not develop any torque, and the retarding output friction force would slow down the motion of the output member. This, in turn, would produce an error.

Thus, the driving force required to sustain the motion of the output member against the retarding output friction force is obtained from the

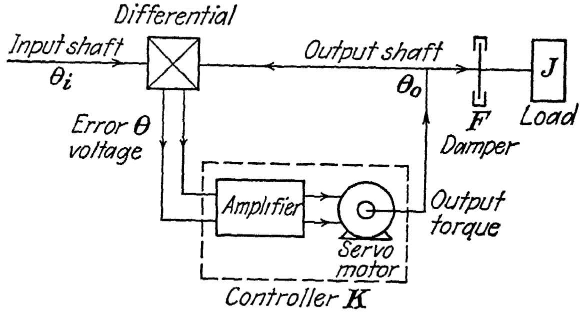

FIG. 4.9.—Electromechanical servomechanism.

error; and these two factors are proportional to each other and necessarily present together in a viscous-damped servo system.

Problem.—In the servo system shown in Fig. 4.10 the output friction coefficient F, measured at the motor shaft, is 50×10^{-6} ft.-lb. per radian per sec. The damping ratio c is equal to 0.25. The inertia J of the motor is 1×10^{-6} slug-ft.^{2}, measured at the motor shaft. The motor shaft is connected to the load through a reduction gear of 100:1 step-down ratio. The moment of inertia of the gears and output load is considered negligible. The motor is energized or controlled by the error voltage through an amplifier. The motor develops a torque T of 0.02 ft.-lb. when a voltage of 100 volts is applied to its terminals. The motor torque being proportional to the applied voltage, and the differential device producing a voltage of 1 volt per deg. error angle θ, determine the required amplifier voltage gain. Determine also the natural frequency ω_n of the system. Calculate the steady-state error θ_S when the input member is driven at a constant speed ω_1 of 20 r.p.m.

Solution: (The reader should attempt to solve the problem before reading the following solution.) Since the factors F, J, and c are given, the controller factor K (torque per unit error angle) can be calculated from the relation

$$F = 2c\sqrt{KJ}, \tag{4.58}$$

which is written

$$\sqrt{K} = \frac{F}{2c\sqrt{J}} = \frac{50 \times 10^{-6}}{2 \times 0.25\sqrt{1 \times 10^{-6}}} = \frac{50 \times 10^{-6}}{0.5 \times 10^{-3}} = 0.1$$

or

$$K = (0.1)^2 = 0.01 \text{ ft.-lb. per radian at the motor shaft.}$$

Since the motor develops a torque of 0 02 ft -lb. for an applied voltage of 100 volts, it will develop a torque of 0.01 ft.-lb. for a voltage of

$$\frac{0.01}{0.02} \times 100 = 50 \text{ volts.}$$

On the other hand, in view of the 100:1 ratio gear train placed between the motor and load, the torque of 0 01 ft.-lb. per radian at the motor shaft corresponds to a displacement of 0.01 radian (or 0.57 deg.) at the output load shaft. And since the differential device is, in effect, driven from the output shaft, the torque of 0.01 ft.-lb. and angular displacement of 0.57 deg. correspond to an error voltage of 0.57 volt The

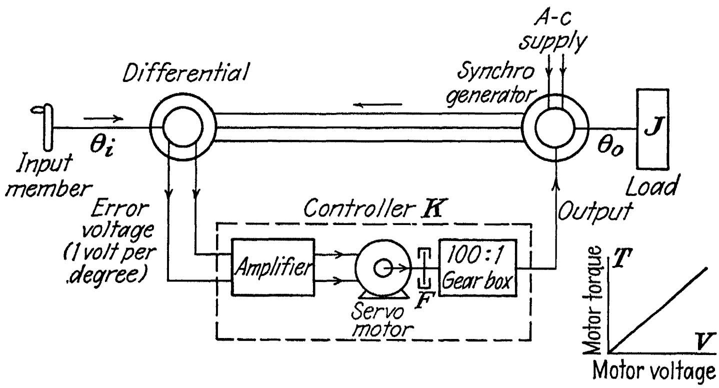

FIG. 4.10.—Geared electromechanical servomechanism.

amplifier gain necessary to obtain the required 50 volts from this 0.57-volt error voltage is then

$$\text{Amplifier gain} = \frac{50}{0.57} = 88.$$

The natural frequency ω_n of the system can be computed from the relation, Eq. (4.50).

$$\omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{0.01}{1 \times 10^{-6}}} = 100 \text{ radians per sec.}$$

In order to calculate the steady-state error, first convert the input speed ω_1 into radians per second.

$$\omega_1 = 20 \text{ r.p.m.} = \frac{20 \times 2\pi}{60} = 2 \text{ radians per sec. (approximate)}$$

The steady-state error θ_s is then obtained from Eq. (4.65). Since it is the steady-state error which is to be found, this implies that sufficient time has elapsed for the transient oscillation to be damped out. The value of time t in the equation being then large, the exponential term becomes negligible, and the equation reduces to

$$\theta_S \frac{\omega_n}{\omega_1} = 2c$$

from which

$$\theta_S = 2c\frac{\omega_1}{\omega_n} = 2 \times 0.25 \times \frac{2}{100} = 0.01 \text{ radian}$$

or

$$\theta_S = 0.01 \times 57 = 0.57 \text{ deg.}$$

Response to Sinusoidal Input Function.—In the preceding discussion a viscous-damped servo control system was subjected to a step input function, whereby the input member of the system was suddenly set in motion at some constant speed. The behavior of the system was then described in terms of certain parameters, such as the damping ratio c and natural frequency ω_n. Expressions were found for the error θ between the positions of the input and output members, as a function of these parameters and the input speed.

In practice, such a step function is not the only type of input function that is of interest in the study of servomechanisms. Nor is it always best suited for experimental measurements of the system parameters c and ω_n. The case will therefore be considered in which the input member is moved back and forth about some average position according to a sinusoidal function of time. The expression of the output position as a function of the input position then not only throws further light on the operating characteristics of the system considered, but also leads to simple methods for determining the constants of the system and adapting the design of the servomechanism to fit definite specified requirements.

The equation of motion of a viscous-damped servomechanism was previously given as

$$K\theta = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt} \tag{4.1}$$

where the error θ was defined

$$\theta = \theta_i - \theta_o. \tag{4.2}$$

By substituting Eq. (4.2) in Eq. (4.1), the differential equation relating the input and output displacements is obtained.

$$K\theta_i = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt} + K\theta_o. \tag{4.66}$$

If now the position of θ_i of the input member is made a sinusoidal function of time of unit amplitude and of radian frequency ω,

$$\theta_i = \cos \omega t, \tag{4.67}$$

the position θ_o of the output member[1] will be a sinusoidal time function of the same frequency, but in general of different amplitude and phase.

$$\theta_o = A \cos (\omega t + \lambda). \tag{4.68}$$

[1] The system is assumed to be linear, that is to say, free from such distortions as may be due to any limiting or saturation. Such a condition is substantially prevalent over a wide range of operating loads, speeds, and frequencies.

In this expression, A is the ratio of the displacement amplitude of the output member to that of the input member, and λ is the phase angle between the input and output member positions.

In accordance with the well-known expansion

$$\epsilon^{j\omega t} = \cos \omega t + j \sin \omega t \tag{4.69}$$

of a complex exponential function into cosine and sine components, the two functions (4.67) and (4.68) may be considered, respectively, as the real parts of the functions

$$\theta_i = \epsilon^{j\omega t} \tag{4.70}$$

$$\theta_o = A\epsilon^{j(\omega t+\lambda)} \tag{4.71}$$

The results of the operations performed below with these functions, Eqs. (4.70) and (4.71), will then apply to the original Eqs. (4.68) and (4.69), if only the real terms of these results be considered. This procedure is followed, according to widespread practice, to simplify the mathematical steps involved.

The first and second time derivatives of the output function, Eq. (4.71), are, respectively,

$$\frac{d\theta_o}{dt} = j\omega A\epsilon^{j(\omega t+\lambda)}, \tag{4.72}$$

$$\frac{d^2\theta_o}{dt^2} = -\omega^2 A\epsilon^{j(\omega t+\lambda)}. \tag{4.73}$$

Substituting Eqs. (4.70), (4.71), (4.72), and (4.73) in Eq. (4.66) gives the equation

$$K\epsilon^{j\omega t} = -\omega^2 JA\epsilon^{j(\omega t+\lambda)} + j\omega FA\epsilon^{j(\omega t+\lambda)} + KA\epsilon^{j(\omega t+\lambda)}. \tag{4.74}$$

Dividing through by $\epsilon^{j\omega t}$, which is never equal to zero, gives

$$K = -\omega^2 JA\epsilon^{j\lambda} + j\omega FA\epsilon^{j\lambda} + KA\epsilon^{j\lambda}, \tag{4.75}$$

from which

$$A\epsilon^{j\lambda} = \frac{K}{K - \omega^2 J + j\omega F}. \tag{4.76}$$

This is a complex expression of the output motion with respect to the input motion. Dividing the numerator and denominator of the right-hand member by K, this expression becomes

$$A\epsilon^{j\lambda} = \frac{1}{1 - \omega^2 \dfrac{J}{K} + j\omega \dfrac{F}{K}}. \tag{4.77}$$

Substituting the values of J/K and F/K, as obtained from Eqs. (4.50) and (4.61),

$$\frac{J}{K} = \frac{1}{\omega_n^2} \qquad \text{and} \qquad \frac{F}{K} = \frac{2c}{\omega_n},$$

Eq. (4.77) is written

$$A\epsilon^{j\lambda} = \frac{1}{1 - \dfrac{\omega^2}{\omega_n^2} + 2jc\dfrac{\omega}{\omega_n}}. \tag{4.78}$$

This can be expressed more simply by introducing a variable d to denote the relative operating frequency, *i.e.*, the ratio of the applied input frequency to the natural frequency of the servo system.

$$d = \frac{\omega}{\omega_n}. \tag{4.79}$$

Equation (4.78) then becomes

$$A\epsilon^{j\lambda} = \frac{1}{1 - d^2 + 2jcd}. \tag{4.80}$$

Equation (4.78), or its equivalent form Eq. (4.80), may be considered as representing a vector of magnitude A and phase angle λ (both relative to the input displacement taken as a unit reference vector), equal, respectively, to

$$\left\{\begin{aligned} A &= \frac{1}{\sqrt{\left[1 - \dfrac{\omega^2}{\omega_n^2}\right]^2 + 4c^2\dfrac{\omega^2}{\omega_n^2}}} = \frac{1}{\sqrt{(1 - d^2)^2 + 4c^2d^2}} && (4.81) \\ \lambda &= -\tan^{-1}\left[\frac{2c\dfrac{\omega}{\omega_n}}{1 - \dfrac{\omega^2}{\omega_n^2}}\right] = -\tan^{-1}\frac{2cd}{1 - d^2}. && (4.82) \end{aligned}\right.$$

For an input function of the form

$$\theta_i = \cos \omega t, \tag{4.67}$$

the output function of the servo system is then obtained by substituting Eqs. (4.81) and (4.82) in Eq. (4.68).

$$\theta_o = \frac{1}{\sqrt{(1 - d^2)^2 + 4c^2d^2}} \cos\left(\omega t - \tan^{-1}\frac{2cd}{1 - d^2}\right). \tag{4.83}$$

Resonance Curves.—Equation (4.83) expresses, as a function of time, the instantaneous position θ_o of the output member of the servomechanism when the input member is being displaced back and forth according

to the sinusoidal time function, Eq. (4.67). Like the input motion, the output motion is a sinusoidal function of time, and both functions have the same frequency ω.

The relative amplitude A of the output member oscillation[1] is expressed in Eq. (4.81) as a function of the damping ratio c and ratio d of the input frequency ω to the natural frequency ω_n of the system. For a given servo system, *i.e.*, for a given value of the parameter c, the output oscillation amplitude is therefore a function of d; hence of the input frequency ω.

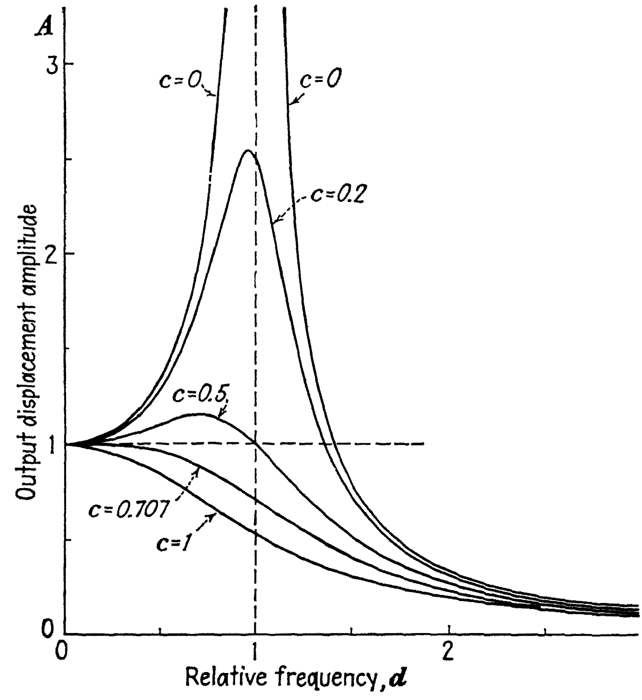

FIG. 4.11.—Resonance curves of servomechanism with viscous output damping, for various values of damping ratio.

This is shown by the curves of Fig. 4.11 for various fixed values of the damping ratio c. When d is small, *i.e.*, when the input member is moved back and forth slowly, the output member oscillates with substantially the same amplitude as the input member. As the input frequency is increased, and if the damping ratio c is small, the output amplitude first increases, passes through a maximum, and finally decreases asymptotically toward zero. Thus, for small values of c, there is a particular input frequency value for which the output member oscillates with maximum

[1] As defined previously, this relative amplitude A is the ratio of the displacement amplitude of the output member to the displacement amplitude of the input member.

amplitude. When the input frequency is increased beyond this value, the output member oscillations become smaller and smaller, and finally the output member remains stationary, its inertia preventing it from moving sufficiently fast to follow the input oscillations.[1]

It will be noted that as the damping ratio c is made larger the maximum output oscillation occurs at an input frequency that is correspondingly smaller. For large values of c the amplitude of the output oscillation decreases continuously as the input frequency is increased, and hence does not pass through a maximum.

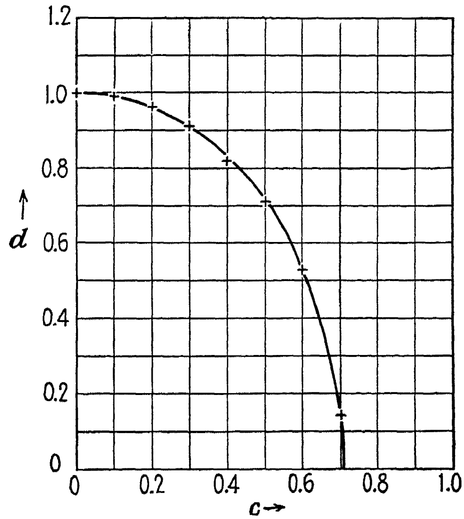

FIG. 4.12.—Position of maximum frequency response as a function of damping ratio for viscous-damped servomechanisms.

The value of d for which the output oscillation amplitude A is a maximum is found by differentiating Eq. (4.81) with respect to d and equating to zero. The value obtained is

$$d_{A\max} = \sqrt{1 - 2c^2} \tag{4.84}$$

for which the maximum amplitude is then

$$A_{\max} = \frac{1}{2c\sqrt{1 - c^2}}. \tag{4.85}$$

Both these expressions have physical significance only for values of c that are smaller than $1/\sqrt{2}$. The two expressions are plotted in the form of curves in Figs. 4.12 and 4.13, respectively.

[1] A discussion of Eq. (4.82) would show corresponding variations of the phase angle λ between the output and input displacements as a function of the relative frequency d, the motion of the output member lagging behind that of the input member by an amount depending on the value of d.

Experimental Measurement of Servo System Parameters.—The preceding results lead immediately to simple laboratory procedures for measuring the parameters of a servomechanism. The input member of the system is displaced back and forth with constant amplitude A_{in} and gradually increasing frequency. The maximum oscillation amplitude

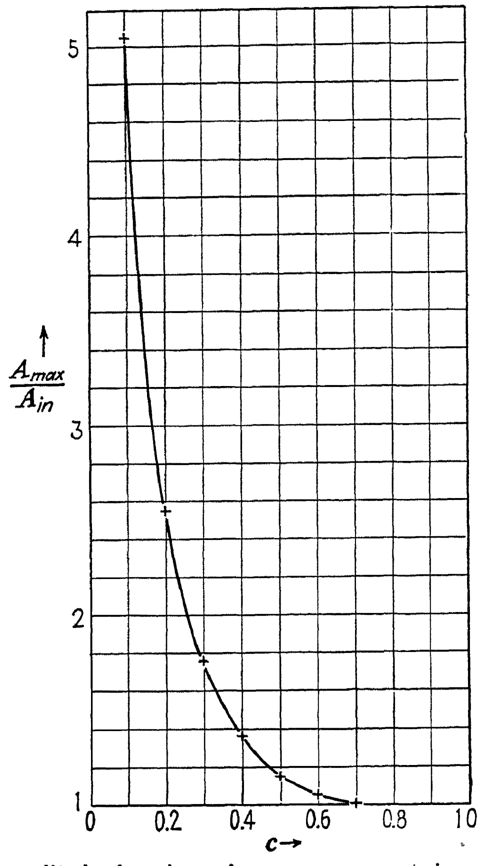

FIG. 4.13.—Ratio of amplitude of maximum frequency response to input signal as a function of damping ratio for viscous-damped servomechanisms.

$A_{out\ max}$ of the output member is measured, as well as the input frequency $\omega_{A\ max}$ at which this maximum occurs.

The value of the damping ratio c may then be read off the graph of Fig. 4.13, corresponding to the measured amplitude ratio $(A_{out\ max})/(A_{in})$, or it may be calculated from Eq. (4.85), written in the form

$$c = \sqrt{\frac{1 - \sqrt{1 - (A_{in}/A_{out\ max})^2}}{2}}. \tag{4.86}$$

On the other hand, the natural frequency ω_n of the system is calculated from the value of c just found and the corresponding value of d, as given by the graph of Fig. 4.12 or by Eq. (4.84). It is equal to

$$\omega_n = \frac{\omega_{A\ \max}}{d_{A\ \max}} \qquad \text{or} \qquad \omega_n = \frac{\omega_{A\ \max}}{\sqrt{1 - 2c^2}}. \tag{4.87}$$

Example.—Suppose it is found experimentally that the maximum output oscillation amplitude is equal to twice the input amplitude, and occurs at an input frequency of 5 cycles per sec.

Then, from the curve of Fig. 4.13,

$$\frac{A_{\max}}{A_{\text{in}}} = 2 \qquad \text{gives} \qquad c = 0.25$$

and the curve of Fig. 4.12 shows

$$\text{for } c = 0.25 \qquad d = 0.94$$

from which

$$\omega_n = \frac{\omega_{A\ \max}}{d_{A\max}} = \frac{5}{0.94} = 5.42 \text{ cycles per sec.}$$

or

$$\omega_n = 5.42 \times 2\pi = 33\ 5 \text{ radians per sec.}$$

Velocity and Acceleration Figures of Merit.—In addition to the parameters and operating factors defined in the preceding paragraphs, which are used for the design and performance calculations of servomechanisms, certain useful *figures of merit* may be defined. The values of these figures of merit allow one to judge quickly the performance quality of a particular servo system and may help in comparing several systems to each other. They also assist in determining what particular parameters of the system may need to be changed in order to improve the performance or keep such performance within specified tolerance limits.

It was found that the operation of a viscous-damped system is expressed by the equation

$$K\theta = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt}, \tag{4.1}$$

which shows that its error θ is the sum of two components, respectively due to the output acceleration $d^2\theta_o/dt^2$ and output velocity $d\theta_o/dt$. There is no error due to the position of the output member, under the conditions considered here.

The *velocity figure of merit* M_ω is then defined as the ratio of the velocity of the output member $d\omega_o/dt$ to the error θ when the output acceleration is zero. Thus, if the acceleration $d^2\theta_o/dt^2$ is set equal to zero, Eq. (4.1) reduces to

$$K\theta = F\frac{d\theta_o}{dt}, \qquad J\frac{d^2\theta_o}{dt^2} = 0, \tag{4.88}$$

which, upon rearrangement and after substituting Eqs. (4.58) and (4.50), gives an expression for the velocity figure of merit.

$$M_\omega = \frac{(d\theta_o/dt)}{\theta} = \frac{K}{F} = \frac{K}{2c\sqrt{KJ}} = \frac{1}{2c}\sqrt{\frac{K}{J}} = \frac{\omega_n}{2c}. \tag{4.89}$$

It follows that for a given output speed the error θ varies inversely as the natural frequency ω_n and directly as the damping ratio c of the system. It should be recalled that reducing the damping ratio in order to improve the velocity figure of merit and reduce the steady-state error will increase the time required for the system to reach steady-state operating conditions.

The *acceleration figure of merit* M_a is defined similarly as the ratio of the acceleration of the output member $d^2\theta_o/dt^2$ to the error θ when the output velocity is zero. Setting this velocity $d\theta_o/dt$ equal to zero in Eq. (4.1), there results

$$K\theta = J\frac{d^2\theta_o}{dt^2}, \qquad F\frac{d\theta_o}{dt} = 0. \tag{4.90}$$

From this, and substituting Eq. (4.50), an expression is obtained for the acceleration figure of merit.

$$M_a = \frac{(d^2\theta_o/dt^2)}{\theta} = \frac{K}{J} = \omega_n^2, \tag{4.91}$$

which shows that for a given output acceleration the error θ varies inversely as the square of the natural frequency ω_n of the system.

Problem.—A viscous-damped servo system has a natural frequency ω_n of 5 cycles per sec. The system possesses only one-quarter of the friction necessary to damp it critically. Determine the velocity and acceleration figures of merit, and calculate the steady-state angular lag of the output member when the input member is driven at a constant speed ω_i equal to 10 r.p.m.

Solution: The natural frequency of 5 cycles per sec. is equal to

$$\omega_n = 5 \times 2\pi = 31.4 \text{ radians per sec.}$$

The damping ratio is given as

$$c = 0.25.$$

Under steady-state operating conditions, the output speed is equal to the constant input speed. Applying the relations, Eqs. (4.91) and (4.89),

$$\text{Acceleration figure of merit} = \omega_n^2 = 31.4^2 = 986$$

$$\text{Velocity figure of merit} = \frac{\omega_n}{2c} = \frac{31.4}{2 \times 0.25} = 62.8.$$

The given input speed ω_i is

$$\omega_i = 10 \text{ r.p.m.} = 10 \times \frac{2\pi}{60} \text{ or } 1.05 \text{ radians per sec.}$$

Then from Eq. (4.89), and noting that $\omega_o = \omega_1$, the steady-state error is

$$\theta = \left(\frac{2c}{\omega_n}\right)\omega_1 = \frac{1}{62.8} \times 1.05 = 0.0167 \text{ radian} = 0.96 \text{ deg.}$$

Geared Servo Control Systems.—In many practical applications of servomechanisms gear trains must be inserted at particular points of the system. In order to calculate the performance of such a geared system as a function of the parameters F, K, J, and in accordance with the methods outlined previously, it is necessary to take into account the transformation ratios of the gear trains by referring all factors involved to some common shaft of the system. This common shaft may, in particular, be either the servo motor shaft or the output load shaft.[1]

All the gear transformations used in a servomechanism can be lumped in the three locations illustrated in the diagram of Fig. 4.14, which shows

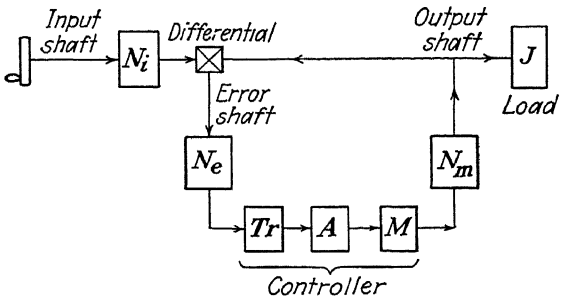

FIG. 4.14.—Possible locations of gear trains in a servomechanism.

a servomechanism of the same general pattern as described before. The system comprises an input shaft, an output shaft and load, a differential device, and a controller. The latter is shown as including a transducer Tr to convert the angular error indicated by the differential into a proportional electrical error voltage; an amplifier A for amplifying this error voltage; and a servo motor M for producing a load-driving torque proportional to the error. Three gear boxes are also incorporated in the system, the respective functions of which are described below.

A step-down gear N_i may be inserted between the input member and the input-output error detecting differential device. A number of turns of the input shaft thus corresponds to one turn of the output shaft. Such an arrangement is used when it is desired to provide a fine (vernier) control of the output load position.

[1] The values of the damping ratio c and natural frequency ω_n, as calculated from the parameters F, K, and J, remain the same, irrespective of what particular shaft these three parameters are referred to. In other words, irrespective of the method of calculation, there is only one value for, respectively, the rate of decay and the frequency of the natural, or free, oscillation of the system.

On the other hand, a step-down gear N_m is placed between the servo motor and output shaft. This combination is used when the motor, which generally is a high-speed device, is required to drive the load at substantially lower speeds than its own operating speed.

Finally, a step-up gear N_e is shown between the differential device and the transducer. This is particularly helpful when some form of synchro generator is used to convert the angular error indication into an equivalent voltage. A displacement of 1 deg. of the output shaft then corresponds to several degrees displacement of the synchro shaft, and this consequently increases the accuracy of the error intelligence controlling the operation of the servo motor.

The functions of the various gear trains, as described above, can be expressed in mathematical terms to obtain the working formulas required for actual design calculations. As stated before, all factors may be referred to the output shaft of the system.

Considering first the controller factor K_o, referred to the output shaft, this is defined as the *output torque* T_o developed at the load divided by the error angle e_o existing at the error shaft of the differential device:

$$K_o = \frac{T_o}{e_o}.$$

Splitting this expression into factors that characterize the individual components of the system illustrated in Fig. 4.14, let these factors be defined as follows:

$$N_e = \frac{\text{transducer error angle}}{\text{output error angle}} = \frac{e_{Tr}}{e_o}$$

$$= \text{error gear ratio (deg. output per deg. input)}$$

$$Tr = \frac{\text{tranducer voltage}}{\text{transducer error angle}} = \frac{V_{Tr}}{e_{Tr}}$$

$$= \text{transducer conversion factor (volts output per deg. input)}$$

$$G = \frac{\text{amplifier output voltage}}{\text{transducer error voltage}} = \frac{V_A}{V_{Tr}}.$$

$$= \text{amplifier gain (volts output per volt input)}$$

$$M = \frac{\text{motor torque}}{\text{amplifier output voltage}} = \frac{T_m}{V_A}$$

$$= \text{motor conversion factor (ft.-lb. torque per volt)}$$

$$N_m = \frac{\text{torque at output shaft}}{\text{motor torque}} = \frac{T_o}{T_m}$$

$$= \text{motor gear ratio (ft.-lb. at load per ft.-lb. at motor shaft)}^1$$

[1] To avoid ambiguity, the torques T_o and T_m, measured at the output and motor shafts, respectively, represent locked torques, as used in Eqs. (4.94) and (4.97). In Eq. (4.95) they represent torques required to oppose the inertia torque, while in Eq. (4.96) they represent torques required to oppose the friction torque.

The relation is then obtained:

$$K_o = N_e \times Tr \times G \times M \times N_m \tag{4.93}$$

$$= \frac{e_{Tr}}{e_o} \times \frac{V_{Tr}}{e_{Tr}} \times \frac{V_A}{V_{Tr}} \times \frac{T_m}{V_A} \times \frac{T_o}{T_m} = \frac{T_o}{e_o} \tag{4.94}$$

as defined before. These relations may be employed for determining one of the component factors as functions of the other factors, when these are known or specified.

Applying now the relation,

$$T = J\alpha \qquad \text{or} \qquad J = \frac{T}{\alpha}$$

given in an earlier chapter as expressing the torque T required to impart an angular acceleration α to a mass having a moment of inertia J, the moment of inertia J_o of the servo motor, referred to the output (load) shaft of the system is expressed

$$J_o = \frac{T_o}{\alpha_o} = \frac{N_m T_m}{\alpha_m / N_m} = \frac{T_m}{\alpha_m} N_m{}^2 = J_m N_m{}^2 \tag{4.95}$$

where N_m is the ratio of the gear inserted between the motor and output shafts, and T_m, α_m, and J_m are, respectively, the motor torque, acceleration, and moment of inertia, as measured at the motor shaft.

Similarly, the friction F_o at the output shaft is expressed in terms of the friction F_m at the motor shaft.

$$F_o = \frac{T_o}{\omega_o} = \frac{N_m T_m}{\omega_m / N_m} = \frac{T_m}{\omega_m} N_m{}^2 = F_m N_m{}^2 \tag{4.96}$$

where ω_o and ω_m are the output and motor speeds, respectively, while T_m is the motor torque, T_o the output torque, and N_m the motor gear ratio, as before.

Finally, the controller factor K_o, measured between the error shaft of the differential at its entrance point into the controller, and the output shaft of the system, is expressed

$$K_o = \frac{T_o}{e_o} = \frac{N_m T_m}{e_m / N_m} = \frac{T_m}{e_m} N_m{}^2 = K_m N_m{}^2, \tag{4.97}$$

where K_m is the torque per unit error, measured at the motor shaft, e_m and e_o represent the error at the motor and output shafts, and T_o, T_m, and N_m have the same meanings as before.

These relations can be summarized in the symbolic equation

$$(K, J, F)_o = N_m{}^2 (K, J, F)_m, \tag{4.98}$$

which expresses the fact that the three factors K, J, and F, referred to the output shaft, are equal, respectively, to the product of the corre-

sponding factor, measured at the motor shaft, by the square of the motor output gear ratio.

Problem.—Determine the effective moment of inertia of a servo system which has a motor whose moment of inertia is 20×10^{-6} slug-ft.^{2}, driving a circular brass disk 1 ft in diameter and ½ in. thick through a gear reduction of 80:1. Assume that the density of brass is 500 lb. per cu. ft.

Solution: The output moment of inertia of the system is the sum of the moment of inertia J_L of the load and the moment of inertia J_m of the motor referred to the output shaft. As set forth in Chap. III, the moment of inertia of the cylindrical disk load is

$$J_L = \frac{MR^2}{2} \tag{3.20}$$

where M and R are the mass and radius of the load.

To compute the mass, first calculate the volume V of the load, that is to say, of a cylinder of radius ½ ft. and thickness $1/24$ ft.

$$V = \pi \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{24} = \frac{\pi}{96} \text{ cu. ft.}$$

The specific mass of the load being 500/32.2 slugs, the mass M of the load is

$$M = \frac{500 \times \pi}{96 \times 32.2} \text{ slugs.}$$

The moment of inertia of the load is then

$$J_L = \frac{500\pi}{96 \times 32.2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = 63{,}500 \times 10^{-6} \text{ slug-ft.}^2$$

The moment of inertia of the motor at the output shaft is equal to the moment of inertia of the motor multiplied by the square of the gear ratio.

$$J_m = 20 \times 10^{-6} \times 80 \times 80 = 128{,}000 \times 10^{-6} \text{ slug-ft.}^2$$

The total moment of inertia of the system is then

$$J_o = J_L + J_m = (63{,}500 + 128{,}000) \times 10^{-6} = 0.1915 \text{ slug-ft.}^2$$

It should be noted that the moment of inertia of the motor (given as 20×10^{-6} slug-ft.^{2}) corresponds to a motor having a rotor of approximately 1 in. diameter, and 1 in. length. The mass of the motor is obviously much smaller than that of the load. Similarly, the motor inertia of 20×10^{-6} slug-ft.^{2} is considerably smaller (less than 1/3,000) than the moment of inertia of the load, which was found to be $63{,}500 \times 10^{-6}$ slug-ft.^{2} Nevertheless, the motor is seen to contribute about twice as much as the load toward the over-all moment of inertia of the servo system. This shows the importance of using a servo motor having as low a moment of inertia as possible. It also illustrates the fact that the load inertia may often be ignored, at least when making a preliminary analysis of a geared servo system.

Problem.—The components of a servo system are such that, when referred to the motor shaft, the moment of inertia J is 5×10^{-6} slug-ft.^{2}, and the friction constant F is 200×10^{-6} ft.-lb. per radian per sec. A friction constant F_c of 500×10^{-6} ft.-lb. per radian per sec. would be necessary to damp the system critically. Calculate the natural frequency ω_n of the servo.

Solution: The damping ratio c of the system is

$$c = \frac{F}{F_c} = \frac{200 \times 10^{-6}}{500 \times 10^{-6}} = 0.4. \tag{4.59}$$

Combining Eqs. (4.50) and (4.58), a relation is then obtained from which the natural frequency may be calculated.

$$\omega_n = \frac{F}{2cJ} = \frac{200 \times 10^{-6}}{2 \times 0.4 \times 5 \times 10^{-6}} = 50 \text{ radians per sec.}$$

or

$$\omega_n = \frac{50}{2\pi} = 8 \text{ cycles per sec.}$$

Problem.—Consider the servo system shown in Fig. 4.15, in which the output load is a disk that is to be driven in position correspondence with a handcrank constituting

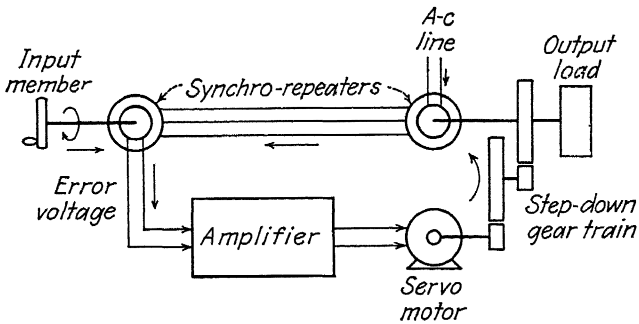

FIG. 4.15.—Geared servomechanism.

the input member. The moment of inertia of the motor and of the load are equal, respectively, to

$$J_m = 1.04 \times 10^{-6} \text{ slug-ft.}^2$$

and

$$J_L = 0.002 \text{ slug-ft.}^2$$

The motor is geared down 100:1 to the load (output) shaft, and the output position is indicated by means of synchro repeaters. These supply to the amplifier a voltage of 1 volt per deg. of input-output error.[1] The moment of inertia of the gears and synchros is disregarded. The motor torque is directly proportional to the voltage applied to the motor terminals and is equal to 0.028 ft.-lb. when this voltage is 115 volts. The friction F at the motor shaft is 50×10^{-6} ft.-lb. per radian per sec., and the damping ratio c of the system is assumed to be 0.25.

Determine the moment of inertia J_o and friction F_o of the system at the output shaft. Find the velocity figure of merit. What is the steady-state error for an input speed of 20 r.p.m. and the required amplifier voltage gain?

Solution: Applying the relation, Eq. (4.95), the moment of inertia J_{mo} of the motor referred to the output shaft is equal to

$$J_{mo} = N_m^2 J_m = 100 \times 100 \times 1.04 \times 10^{-6} = 0\,0104 \text{ slug-ft.}^2.$$

[1] In practice this condition is approximately true when the error angle does not exceed a few degrees, as stated previously.

The total moment of inertia J_o of the system, referred to the output shaft, is then

$$J_o = J_{mo} + J_L = 0.0104 + 0.002 = 0.0124 \text{ slug-ft.}^2$$

Similarly, from the relation Eq. (4.96), the friction F_o at the output shaft is

$$F_o = N_m{}^2F_m = 100 \times 100 \times 50 \times 10^{-6} = 0.5 \text{ ft.-lb. per radian per sec.}$$

Combining Eqs. (4.50) and (4.58), the natural frequency ω_n of the system is

$$\omega_n = \frac{F_o}{2cJ_o} = \frac{0.5}{2 \times 0.25 \times 0.0124} = 80.6 \text{ radians per sec.}$$

The velocity figure of merit is then (from Eq. 4.89)

$$M_\omega = \frac{\omega_n}{2c} = \frac{80.6}{2 \times 0.25} = 161.2.$$

The steady-state error θ is obtained from the relation, Eq. (4.89).

$$\theta = \frac{\omega_o}{M_\omega} = \frac{\omega_1}{M_\omega}$$

in which

$$\begin{aligned}\omega_1 &= 20 \text{ r.p.m.}\\ &= \frac{20 \times 360}{60} \text{ or } 120 \text{ deg. per sec.}\end{aligned}$$

so that

$$\theta = \frac{120}{161.2} = 0.745 \text{ deg.}$$

On the other hand, from the relation, Eq. (4.50), the torque at the output shaft is calculated

$$K_o = \omega_n{}^2J_o = 80.6^2 \times 0.0124 = 80.5 \text{ ft.-lb. per radian error.}$$

Then, noting that the error voltage from the synchro repeaters is 1 volt per deg. error, or a rate of 57.3 volts per radian error, Eq. (4.93) can be written

$$K_o = 80.5 = 1 \times 57.3 \times G \times \frac{0.028}{115} \times 100,$$

from which the required amplifier voltage gain G is obtained.

$$G = \frac{80.5 \times 115}{57.3 \times 0.028 \times 100} = 57.7.$$

As a check of the preceding calculations, it will be observed that different methods may now be used for calculating the motor torque. Referring first to Eq. (4.9),

$$K\theta = F\omega_1,$$

which states that the motor torque $K\theta$ must equal the friction drag $F\omega_1$, under steady-state condition, these quantities become

$K\theta = 80.5 \times \frac{0.745}{57.3} = 1.04$ ft.-lb. (note that the error of 0.745 deg. is here converted into radians by dividing by 57.3).

$F\omega_1 = 0.5 \times 20 \times \frac{2\pi}{60} = 1.04$ ft.-lb., which agrees with the value just found (the input speed of 20 r.p.m. is converted into radians per sec. by multiplying it by $2\pi/60$).

Finally, since the error was found to be 0.745 deg., the amplifier input voltage (error voltage) is 0.745 volt. The calculated amplifier gain is 57.7, and the resulting voltage applied to the servo motor is

$$0.745 \times 57.7 = 43 \text{ volts.}$$

And since the motor torque is proportional to the applied motor voltage,

$$\frac{0.028}{115} = \frac{T}{43}$$

from which

$$T = \frac{0.028 \times 43}{115} = 0.0104 \text{ ft.-lb.}$$

at the motor shaft, or 1.04 ft.-lb. at the output shaft, taking into account the 100:1 step-up motor gear ratio. This, too, agrees with the two values calculated before.

Various Forms of Viscous Damping.—Viscous damping, as defined in the preceding discussion, is caused by a force proportional to the output speed of motion of the system directed opposite to the output speed. The useful function of such damping is to reduce the amplitude and duration of the transient oscillation, which occurs whenever the input speed is changed.

Viscous damping can be obtained through mechanical friction devices such as dashpots, friction disks, and the like, linked to the output member and adjustable by mechanical means. Electromagnetic friction devices may also be used, such as an eddy-current damper, which consists of a metal disk rotating in a magnetic field. The disk may be the rotor of an a-c servo motor, and the magnetic field is then produced by a direct current that is made to flow through the motor field winding, in addition to whatever alternating current the motor may require for its normal operation. The damping can be set to a proper value by adjusting the intensity of the direct current in the field winding.

In all these devices a constant torque must be applied by the motor to the output member to counterbalance the opposing friction torque, if a constant output speed is to be maintained. The driving and friction torques must be of equal magnitudes. Under this condition, energy drawn from the motor is directly transformed into heat in the friction damper.

This continuous energy consumption can be avoided, and more economical and efficient operation obtained through the use of other viscous-damping methods. However, before discussing these methods, it is necessary to understand thoroughly the process of viscous damping itself.

Consider again the idealized servo control system shown in Fig. 4.15. Under steady-state operating conditions, the system is subjected to two forces only, *viz.*, the driving torque developed by the servo motor and the opposing retarding torque due to the friction of the system.

The driving torque T produced by the motor is, as explained before, proportional to the input-output position error θ.

$$T = K\theta \tag{4.99}$$

where K, the controller factor, is the motor torque per unit error angle (foot-pounds per radian error). In this idealized system, the torque T is entirely independent of the speed of motion and depends solely on the error voltage, the voltage amplifier gain, and the voltage-torque characteristic of the motor.

It follows that the speed-torque performance of the motor can be represented by a family of curves, each curve corresponding to some value

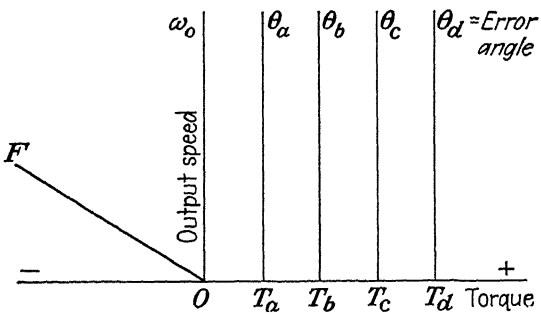

FIG. 4.16.—Motor torque and friction torque vs. speed.

of error angle or error voltage. This is illustrated in Fig. 4.16, where constant values T_a, T_b T_c, . . . of motor torque are represented by straight lines parallel to the vertical speed axis. These torque values are independent of the speed ω_o, and correspond, respectively, to given values θ_a, θ_b, θ_c, . . . of the error angle.[1] If the motor torque increases linearly with the applied motor voltage, it also increases linearly with the error angle θ, since this determines the error voltage.

On the other hand, the retarding friction torque is independent of the error and equal to the product $F\omega_o$ of the friction coefficient F and the output speed ω_o. This is shown in Fig. 4.16 by the straight line F, the friction torque being plotted negatively[2] since it is directed opposite to the speed, or direction of motion of the output member.

[1] The motor torque $T = K\theta$ will be positive or negative, depending, respectively, on whether the error θ is positive or negative.

[2] Equation (4.1) can be written

$$K\theta - F\frac{d\theta_o}{dt} = J\frac{d^2\theta_o}{dt^2} \tag{4.1a}$$

in which $K\theta$ is the motor torque, θ being the error angle, while $F(d\theta_o/dt)$ is the friction torque and $d\theta_o/dt$ is the output speed, designated as ω_o in the text. The negative polarity of the friction torque is apparent immediately in the equation.

Let then the diagram of Fig. 4.17 represent the conditions for one given value θ_a of the input-output error. The straight line M represents the motor torque T_a corresponding to this error value. Since the torque is independent of the speed, the line is parallel to the speed axis. The line F represents the friction torque as before. The line S is a composite of the two lines M and F, which is obtained by adding the torque contributions from the motor and friction at each output speed. The line S, therefore, represents the over-all speed-torque characteristic of the motor and damping for the particular error value θ_a. This line thus may be interpreted as representing the resultant torque that is applied to the inertia load of the system at various output speeds, as expressed by Eq. (4.1a) in the preceding footnote.

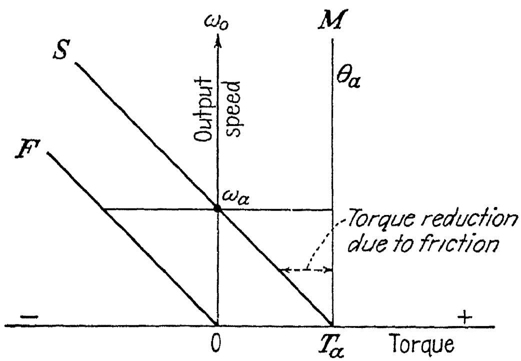

FIG. 4.17.—Motor torque and friction torque combined.

The value of the output speed ω_o where the line S crosses the speed axis (zero torque line), which will be designated as ω_a, must then be the steady-state speed at which the output member will run when the error is equal to θ_a, because at that point there is no torque either to accelerate or to decelerate the inertia load. As shown by the line S, any further increase of the output speed causes the torque to become negative, which decelerates the system. Any decrease in speed causes the torque to become positive, and this, in turn, accelerates the system. Thus, the speed tends to become stabilized at this value ω_a.

Conversely, the constant error θ_a corresponds to a steady-state operating condition of the system where the input and output members have some constant speed ω_a. However, this same error can be obtained by locking the output member while the input member, instead of being driven continuously at the speed ω_a, is simply displaced by the angle θ_a. The system being then motionless, the friction torque is zero, and the motor torque is equal to T_a, the locked motor torque for an applied voltage corresponding to the error θ_a.

The force due to friction was previously given as equal to the product $F\omega_o$. This may be written

$$F = \frac{T}{\omega_o}, \tag{4.100}$$

where T is the friction torque corresponding to an output speed ω_o. On Fig. 4.17 the friction torque at the output speed ω_a is equal to the motor torque T_a, and therefore by substituting these values for ω_o and T in the preceding equation, the relation is obtained

$$F = \frac{T_a}{\omega_a}, \tag{4.101}$$

where ω_a is the operating speed at which the error is equal to θ_a, and T_a is the locked torque of the motor with an applied motor voltage equal to

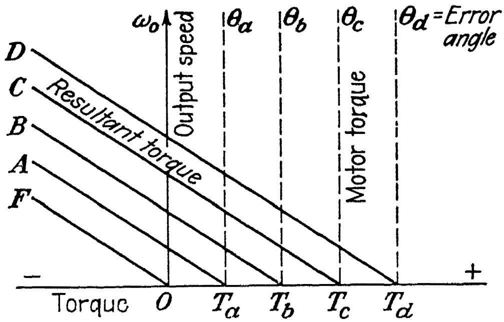

FIG. 4.18.—Torque-speed characteristic of system.

that corresponding to an error θ_a. More generally, the friction F is equal to the slope $-\Delta T/\Delta\omega_o$ of the speed-torque characteristic, with respect to the vertical axis (speed axis).

Generalizing the preceding explanation, the total or net torque operating on the inertia load of the system at all times (*i.e.*, during transient as well as steady-state conditions) is equal to the algebraic sum of the friction torque corresponding to the output speed considered and the motor torque corresponding to the error. The speed-torque performance of the system is therefore represented by a set of straight lines, parallel to the line F, each of which corresponds to some particular value of the error θ. For example, the lines A, B, and C shown in Fig. 4.18 correspond, respectively, to error values θ_a, θ_b, and θ_c.

Frictionless Viscous Damping.—Viscous damping, as obtained by means of some friction device, was shown in the preceding paragraph to result in a speed-torque characteristic of the system that is represented graphically by a set of negatively sloping lines cutting the torque and speed axes in their positive regions. Any servo system having such a

speed-torque characteristic will have the same operating properties as a friction-damped system, as discussed in earlier parts of this chapter. Viscous friction, produced in mechanical or electromagnetic friction dampers, is only one of several means of developing this viscous damping speed-torque characteristic. Other methods are described below.

As a first example, consider the servo control system shown in Fig. 4.19. It differs from the system of Fig. 4.1 in that, instead of a friction damper, a small generator or tachometer S is coupled to the output shaft. The voltage produced by this generator is proportional to the rotational speed of the output shaft and load, and is combined with the error signal in such a polarity as to reduce the voltage applied to the motor. The error voltage is obtained, as usual, from the input-output differential device through a transducer T.

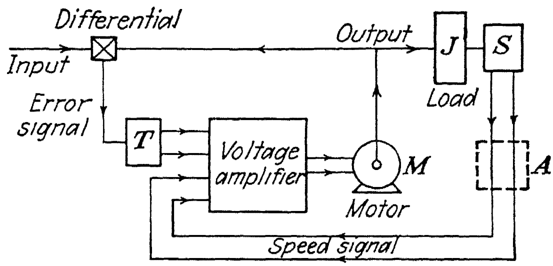

FIG. 4.19.—Servomechanism with damping through speed signal feedback.

Since the motor torque is proportional to the applied motor voltage, which in turn is the amplified combination error and speed voltage, it follows that for a constant error the motor torque decreases when the output speed increases. This is precisely the condition required for obtaining speed-torque curves similar to the viscous damping characteristics shown in Figs. 4.16, 4.17, and 4.18. In some cases, an auxiliary amplifier, shown in dotted lines in Fig. 4.19, may be inserted between the tachometer S and error voltage amplifier.

Although the method illustrated here produces the same damping effect as that obtained with mechanical friction, it has certain advantages over the latter. It was shown that an increased friction coefficient reduces the transient oscillation. It was shown also that in the steady-state operating condition the motor torque is exactly counterbalanced by the retarding friction torque. The available motor torque therefore sets a limit to the amount of friction damping that may be applied to the system, while leaving a fraction of this torque to perform useful work in the connected output load. On the other hand, in the electrically damped system shown in Fig. 4.19, the equivalent damping, as represented by the slope of the speed-torque curves, absorbs no energy from

the motor and may be made as large as desired by suitably amplifying the speed signal voltage obtained from the tachometer S.

Instead of coupling a separate generator to the output shaft, as shown in the preceding example, the counterelectromotive force generated by the servo motor itself may be fed back into the controller amplifier, as a speed-proportional voltage, to vary the amplifier output.

A method utilizing this principle is illustrated in Fig. 4.20. In this particular instance, the servo motor is a two-phase (split-phase) induction motor. One phase is permanently excited by the 115-volt supply line, while the other is fed by a quadrature error voltage through the controller amplifier. The torque of the motor is proportional to this error voltage. The variable-voltage winding ϕ_1 of the motor is connected as one arm of a

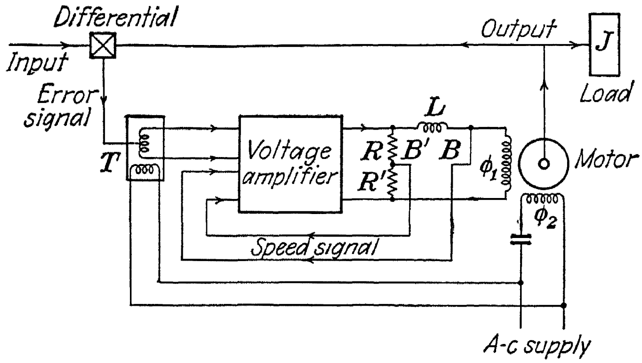

FIG. 4.20.—Servomechanism with damping through speed signal feedback.

bridge network, the other arms of which are, respectively, an inductance coil L and two resistors R and R'. These are so proportioned that no voltage appears at the bridge terminals B and B' when the motor is at rest. When the motor rotates, a voltage is developed at these terminals, which is proportional to the motor speed. This voltage is then applied to the amplifier input terminals, in opposition to the error voltage, so that the total signal voltage applied to the amplifier decreases as the motor speed increases. Here again, therefore, the torques due to error and output speed are oppositely directed, producing a sloping speed-torque characteristic curve similar to the curves shown in Figs. 4.16, 4.17, and 4.18.

Viscous Damping through Motor Characteristic.—Instead of using an output-speed signal to reduce the error voltage and produce the typical speed-torque curve characteristic of a viscous-damped system, a servo motor may be used that has an inherent performance characteristic that produces this same result. The required properties are found, in particular, in certain types of induction motors, as will be discussed below.

A usual type of two-phase induction motor is illustrated schematically in Fig. 4.21. The stator is provided with two similar windings ϕ_1 and ϕ_2 generally distributed in slots around the stator circumference, having their magnetic axes at right angles to each other. The rotor R may be a wire-wound coil short-circuited upon itself, or it may consist of the well-known *squirrel cage* structure comprising a number of longitudinal conductors shorted at both ends by end rings or plates.

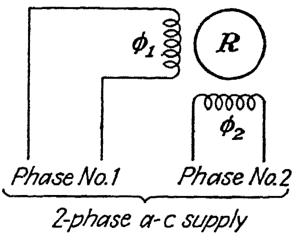

FIG. 4.21.—Two-phase induction motor.

The two stator windings are excited by alternating voltages which are electrically displaced by 90 deg. with respect to each other. The resulting magnetic field set up in the machine then rotates at the rate of one revolution during every cycle of the exciting voltage. The rotor is not connected to any external circuit, but the current induced in it by the rotating magnetic field of the stator coils causes it to follow the rotation of this field. Reversing the phase of either of the two stator voltages (by 180 deg.) reverses the direction of rotation of the magnetic field and of the motor torque.

The quadrature alternating voltages required for exciting the stator coils may be obtained, as shown in Fig. 4.21, by connecting these coils to the separate phase terminals of a two-phase a-c supply. When only a single phase supply is available, both windings may be connected, in

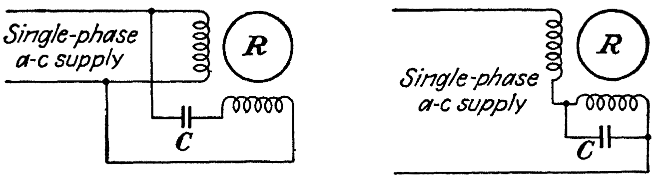

FIG. 4.22.—Split-phase induction motor.

parallel or in series, across this supply, as shown in Fig. 4.22. Capacitors C are then connected in series or in parallel with one of the stator windings, to shift by approximately 90 deg. the relative phase of the currents flowing in the windings.

The speed-torque characteristic of such a two-phase induction motor is largely determined by the electrical resistance of its rotor winding. This is shown in Fig. 4.23, in which, for a given applied voltage value, curves 1, 2, and 3 correspond, respectively, to a low, medium, and high resistance value of the rotor winding. In the case of high-resistance value, the torque decreases as the speed increases, giving a characteristic that is similar to that of a viscous-damped system, as discussed in the preceding paragraphs.

A servo control system employing an induction motor that has a high-resistance rotor then assumes the simple form shown in Fig. 4.24. This is substantially similar to the system illustrated in Fig. 4.1, except that the friction damper is omitted. The differential device operates a transducer, which furnishes an alternating voltage of supply line fre-

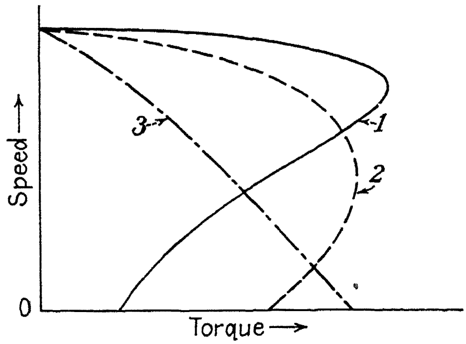

FIG. 4 23.—Speed-torque characteristics of induction motors.

quency and of amplitude that is proportional to the input-output error. The error voltage is amplified, and then applied to one of the two stator windings of the servo motor. The other stator winding of this motor is energized directly from the a-c supply line. A capacitor is inserted in the circuit of one of the two stator windings to provide the required

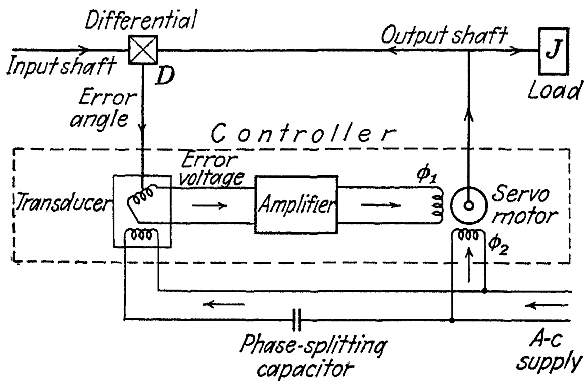

FIG. 4.24.—Servomechanism with two-phase induction motor.

90-deg. phase shift between the two stator currents. When the error reverses direction, the error voltage reverses phase 180 deg., which causes the motor torque to reverse.

Under these conditions, the speed-torque characteristic of the system will be substantially a straight line, as shown in 3, Fig. 4.23. The actual position of this line with respect to the axes of coordinates depends on

the voltage applied to the control phase ϕ_1 of the motor, the torque and speed being directly proportional to this voltage. This is shown in Fig. 4.25, where different error voltages V_1, V_2, V_3, . . . differing by equal amounts, cause the speed-torque characteristic to be shifted by correspondingly equal amounts, while remaining parallel to itself.

A family of parallel lines is thus obtained, similar to the lines A, B, C, . . . of Fig. 4.18. They exhibit the fundamental property of a viscous-damped system, whereby the retarding force, or reduction of available driving torque, increases directly as the operating speed. However, in the case illustrated in Fig. 4.25, the curves apply only for speed values smaller than the synchronous speed of the induction motor. For a given servo system, a motor is then chosen for which the maximum steady-state operating speed is approximately 55 per cent of synchronous

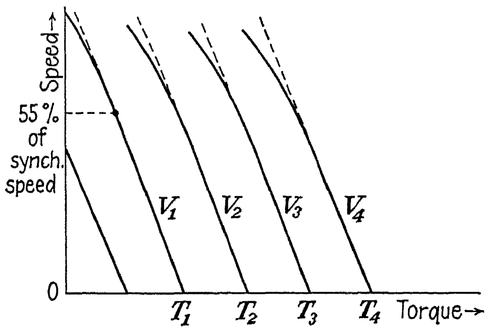

FIG. 4.25.—Speed-torque characteristics of two-phase induction motor with one variable-voltage phase winding.

speed. This is in order that, under varying speed conditions, the operating point of the system may remain on the straight sloping portion of the speed-torque characteristic.

This condition being kept in mind, the speed-torque characteristics may, for the purpose of calculations, be extended as straight lines from the torque axis to the speed axis, as shown by the dotted lines in Fig. 4.25.

Damping Coefficient of Induction Motor.—It was explained in relation to Figs. 4.17 and 4.18 that the friction coefficient is equal to the ratio $-\Delta T/\Delta S$ of corresponding torque and speed variations. This ratio is the reciprocal of the slope of the torque-speed curves shown in the diagram. In other words, it is the rate at which, due to viscous friction or its equivalent, the torque decreases when the speed increases. This relation applies, irrespective of what method is used to obtain viscous damping.

Thus, let Fig. 4.26 represent the torque characteristics, measured experimentally, of a two-phase induction motor having one phase con-

stantly excited and the other phase variably fed. On the assumption that there is no other source of friction in the system, or that any friction due to other sources is negligibly small, the friction coefficient of the system is then equal to

$$F = \frac{\text{torque retardation}}{\text{change in speed}} = \frac{-\Delta T}{\Delta S} = \frac{0.02 \text{ ft.-lb.}}{1{,}000 \text{ r.p.m.}}$$

and since

$$1{,}000 \text{ r.p.m.} = 1{,}000 \times \frac{2\pi}{60} \text{ radians per sec.,}$$

it follows that

$$F = \frac{0.02}{1{,}000} \times \frac{60}{2\pi} = 191 \times 10^{-6} \text{ ft.-lb. per radian per sec.}$$

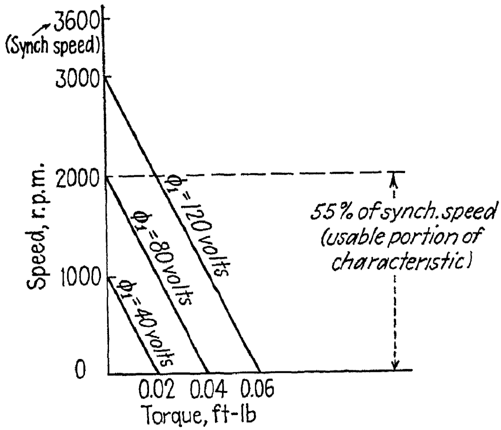

FIG. 4.26.—Speed-torque characteristics of two-phase induction motor with one variable-voltage phase winding.

Problem.—A viscous-damped servo system driven by a motor with characteristics similar to those illustrated in Fig. 4.26 is found to have a steady-state output lag angle of 2 deg. when the output load is driven at a speed $\omega_1 = 20$ r.p.m. The motor drives the output load through a gear reduction of 100:1 ratio. If a synchro follow-up link is used that produces approximately 1 volt for a 1-deg. difference between the input and output shaft positions, determine the amplifier gain required to drive the motor.

Solution: Under steady-state operating conditions, the load is neither accelerated nor decelerated:[1] the driving motor torque, equal to the product $K\theta$ of the controller gain by the error angle, then equals the retarding (friction) torque $F\omega_1$, or product of the effective friction coefficient by the output speed. Thus

$$K\theta = F\omega_1$$

from which

$$K = \frac{F\omega_1}{\theta}$$

[1] According to Newton's law, the net torque acting on a body rotating at constant speed is equal to zero.

which expresses the controller gain (torque per radian error) as a function of the friction coefficient, output speed, and steady-state error.

The coefficient F was found in the preceding example to be

$$F = 191 \times 10^{-6} \text{ ft-lb. per radian per sec. measured at the motor shaft.}$$

The speed is given as

$$\omega_1 = 20 \text{ r.p.m., or } 20 \times \frac{2\pi}{60} \text{ radians per sec.}$$

The error is also given.

$$\theta = 2 \text{ deg., or } \frac{2 \times 2\pi}{360} \text{ radians.}$$

Taking into account the reduction gear ratio of 100:1, as shown in the relation, Eq. (4.97), the factor K is calculated

$$K = 191 \times 10^{-6} \times 20 \times \frac{2\pi}{60} \times \frac{360}{2 \times 2\pi} \times 100^2 = 115.$$

On the other hand,

$$K = \frac{\text{error volts}}{\text{radians error}} \times \text{amplifier gain} \times \frac{\text{motor torque}}{\text{motor volts}} \times \text{gear ratio}$$

in which $K = 115$, as just calculated,

$$\frac{\text{Error volts}}{\text{Radians error}} = 57.3, \text{ since the synchro follow-up link produces 1 volt per deg. error,}$$

$$\frac{\text{Motor torque}}{\text{Motor volts}} = \frac{0.06}{120} \text{ from the motor curves, Fig. (4.26),}$$

$$\text{Gear ratio} = 100.$$

From this, it follows

$$\text{Amplifier gain} = 115 \times \frac{1}{57.3} \times \frac{120}{0.06} \times \frac{1}{100} = 40.$$

The same result may be arrived at in a shorter way as follows: since the output speed is 20 r.p.m. and the motor reduction gear has a ratio of 100:1, the motor speed is 2,000 r.p m. As this is the steady-state operating speed, the load is neither accelerated nor decelerated, so that the effective torque in the system is zero. From the diagram of Fig. 4.26 it is found that a motor speed of 2,000 r.p.m. at zero torque corresponds to a motor voltage of 80 volts. The steady-state error is given as 2 deg., which produces an error voltage of approximately 2 volts at the amplifier input terminals. In order to obtain from the amplifier the required motor voltage of 80 volts, the amplifier must then have a gain of $^{80}\!/_{2} = 40$, which agrees with the preceding computation.

Torque/Inertia Figure of Merit.—An important factor in the performance quality of a position-control servomechanism is the liveliness with which it responds to speed variations of its input member. The torque/inertia ratio of a controller may serve as a good indication of the potential performance of the servomechanism into which it is to be incor-

porated. However, the measure in which an increased controller torque/inertia ratio may improve the operation of a given servo depends on the particular application for which the mechanism is intended. Broadly speaking, the controller must be so chosen as to match properly the characteristics of the load.

In a general way, two types of servo applications may be considered here:

1. The controller may be required to supply torque to an energy-consuming external load having a comparatively small moment of inertia. The torque capacity of the controller motor then need not substantially exceed the torque necessary for moving the output load over the desired range of speeds.

2. The controller torque may be applied to an output member that has a large moment of inertia, while the output energy consumption is sufficiently small to be negligible. It is for this type of application that the system torque/inertia ratio may usefully be considered as a convenient *maximum acceleration* figure of merit.

This may readily be understood by recalling that, as described in Chap. III, a torque T, applied to a load having a moment of inertia J, imparts to the load an angular acceleration α such that

$$T = J\alpha. \tag{3.13}$$

Written under the form

$$\alpha = \frac{T}{J} \tag{4.102}$$

this relation shows that the acceleration is directly proportional to the torque/inertia ratio of the system.

When this torque/inertia ratio is considered, the entire inertia of the system must be taken into account, the various inertia components (motor, gears, output load) being, for this purpose, referred to a common shaft. The motor torque must then be referred to this same shaft.

This figure of merit is particularly important when the system is subject to frequent changes of speed. It complements the indication given by the acceleration figure of merit previously defined (Eq. 4.91), which is specifically valid for zero output velocity.

Since the heat dissipation of the controller motor is limited, it is often preferable to design the controller in such manner that limiting will occur in the amplifier rather than in the motor, in order to avoid overloading the latter.

Problem.—Two 60-cycle, two-phase induction motors[1] are designed to deliver 63 per cent of their locked torque at 55 per cent synchronous speed (*i.e.*, 55 per cent of

[1] Diehl Manufacturing Company, Finderne, N.J.

3,600 r.p m.), when driven from a two-phase supply. These motors have the following approximate characteristics:

Motor	Nominal rating, watts	Inertia, slug-ft.^{2}	Locked torque, ft -lb.	Torque/inertia ratio, T/J
8	1.5	0.28×10^{-6}	0.013	49,000
25	5	1.04×10^{-6}	0.028	27,000

Determine which one of these motors will furnish the best performance in a servo system where the load has a moment of inertia $J_L = 0.008$ slug-ft.^{2} and is to be driven at a speed of 20 r.p m.

Solution: The normal motor operating speed being $3{,}600 \times 0.55 = 2{,}000$ r.p.m. approximately, a 100:1 reduction gear will be inserted between the motor and load in order to obtain the specified load speed of 20 r.p.m. Referring the moment of inertia of the load to the motor shaft, the torque/inertia ratio of the servo system is then calculated from the relation

$$\left(\frac{T}{J}\right)_{\text{servo}} = \frac{T_{\text{motor}}}{J_{\text{motor}} + (J_{\text{load}}/N^2)} = \left(\frac{T}{J}\right)_{\text{motor}} \times \frac{J_{\text{motor}}}{J_{\text{motor}} + (J_{\text{load}}/N^2)} \quad (4.103)$$

where $N = 100$ is the ratio of the motor output gear.

Substituting the known values, this relation becomes

$$\text{with motor 8: } \left(\frac{T}{J}\right)_{\text{servo}} = 49{,}000 \times \frac{0.28}{0.28 + 0.8} = 12{,}700$$

$$\text{with motor 25: } \left(\frac{T}{J}\right)_{\text{servo}} = 27{,}000 \times \frac{1.04}{1.04 + 0.8} = 15{,}300.$$

Thus, although the motor 25 itself has a smaller torque/inertia ratio than the motor 8, it gives a better (higher) torque/inertia ratio to the over-all system.

Static Loading.—In the preceding discussion of servomechanisms it was assumed that the total load of the system is proportional to the output speed, as is the case for pure viscous output loading. In practice, this is not always the case, either due to the presence of a certain amount of coulomb friction or to some constant loading that is independent of the operating speed of the system. This must be taken into consideration when the required motor size and other parameters of the mechanism are calculated, as illustrated in the following example.

Problem.—A small asymmetrical radio antenna weighing 6 lb. and having a radius of gyration of 6 in. is to be rotated at a speed of 20 r.p.m. with a steady-state error of less than 1.5 deg. (0.026 radian). It is subjected to a wind load that creates a retarding torque of 5 ft.-lb. Calculate the elements of the servo, knowing that the following two-phase induction motors[1] are available. These motors are all designed for 60-cycle operation and deliver 63 per cent of locked torque at 55 per cent of synchronous speed.

[1] Diehl Manufacturing Company, Finderne, N.J.

Motor	Nominal rating, watts	Inertia, slug-ft.^{2}	Locked torque, ft.-lb.	Torque/inertia ratio, T/J
49	25	8.9×10^{-6}	0.151	17,350
66	60	14.8×10^{-6}	0.336	23,200
84	100	27.0×10^{-6}	0.560	20,700

Solution: Taking the normal running speed of the motor as 2,000 r.p.m., which is approximately equal to 55 per cent of its synchronous speed ($3{,}600 \times 0.55 = 2{,}000$ approximately), the required output speed of 20 r.p.m. will be obtained through a 100:1 reduction gear.

Designating by M and R the mass and radius of gyration of the load, the moment of inertia of the load is equal to

$$J_L = MR^2 = \frac{6}{32.2} \times \frac{1}{2} \times \frac{1}{2} = 0.047 \text{ slug-ft.}^2$$

Referred to the motor shaft, this load moment of inertia is equal to

$$J_{LM} = 0\ 047 \times \frac{1}{100^2} = 4.7 \times 10^{-6}.$$

The total moment of inertia and torque/inertia ratio of the system, with each of the three motors, are then, respectively,

Motor	Nominal rating, watts	Moment of inertia of system, slug-ft.^{2}	Torque/inertia of system
49	25	$(8.9 + 4.7) \times 10^{-6} = 13.6 \times 10^{-6}$	11,000
66	60	$(14.8 + 4.7) \times 10^{-6} = 19.5 \times 10^{-6}$	17,200
84	100	$(27.0 + 4.7) \times 10^{-6} = 31.7 \times 10^{-6}$	17,800

This shows at once that motor 49, in view of the smaller torque/inertia ratio of the system, is not as suitable as the two other motors. Since these last two motors give torque/inertia ratios of the same order of magnitude, the smaller one (66) may be found the most practical, as it is about half the size of motor 84 (60 watts against 100 watts).

Using motor 66, the friction damping afforded by its sloping speed-torque characteristic is

$$F_M = \frac{\Delta T}{\Delta S} = \frac{(100 - 63) \times 0.336}{100} \times \frac{1}{3{,}600 \times 55/100} \times \frac{60}{2\pi}$$
$$= 0.0006 \text{ ft.-lb. per radian per sec.}$$

Referring this and the moment of inertia of the system to the output shaft, these become

$$F_0 = 0.0006 \times 100^2 = 6 \text{ ft.-lb. per radian per sec.}$$
$$J_0 = 19.5 \times 10^{-6} \times 100^2 = 0.195 \text{ slug-ft.}^2$$

To calculate the controller gain K, the steady-state operating relation is applied

$$K\theta = F\omega_1,$$

where
$$\omega_1 = 20 \text{ r.p.m.} = 20 \times \frac{2\pi}{60} \text{ radians per sec.}$$

$$\theta = 1.5 \text{ deg.} = \frac{1.5}{57.3} \text{ radian.}$$

Then

$$K = \frac{F\omega_1}{\theta} = 6 \times 20 \times \frac{2\pi}{60} \times \frac{57.3}{1.5} = 480.$$

The natural frequency of the system is

$$\omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{480}{0.195}} = \sqrt{2{,}500} = 50 \text{ radians per sec.}$$

Applying Eq. (4.65) after the steady-state operating condition has been reached, the damping ratio c is determined.

$$\theta\frac{\omega_n}{\omega_1} = 2c$$

or

$$c = \theta\frac{\omega_n}{2\omega_1} = \frac{1.5}{57.3} \times \frac{50}{2 \times 20} \times \frac{60}{2\pi} = 0.31.$$

From the curve of Fig. 4.7 it is found that for this value of the damping ratio the maximum swing of the transient oscillation is 1.9 times as great as the steady-state

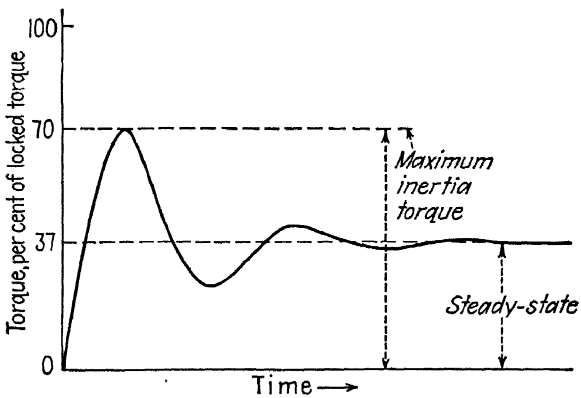

FIG. 4.27.—Torque variations in servomechanism discussed in problem.

error. Hence, the maximum torque required from the motor for the transient oscillation is also 1.9 times the torque necessary to drive the load under steady-state conditions at 20 r.p.m., not taking into account the constant wind load of 5 ft.-lb.

It will be recalled that, under full excitation, the motor delivers 63 per cent of locked torque when it is running at 55 per cent of synchronous speed. Referring back to Fig. 4.25, let the torque-speed curve V_4 represent this full-voltage operation of the motor. Assuming that the torque-speed curves for various excitation voltages of the motor are straight, parallel lines, it is seen that the line that corresponds to a voltage for which the locked torque is 37 per cent of the full-voltage locked torque will cut the speed axis at the point of 55 per cent of synchronous speed. For this

voltage and at that speed, the torque is therefore zero. In other words, the motor then produces neither an accelerating nor a decelerating torque on the load, and a steady-state condition prevails.

Since the steady-state torque is 37 per cent of the locked torque, the maximum transient torque required to accelerate the system against its inertia torque is

$$1.9 \times 0.37 = 0.7 \text{ of locked torque.}$$

The wind load of 5 ft.-lb. when referred to the motor shaft becomes equal to 0.05 ft.-lb. The locked torque of the motor being 0.336 ft.-lb., this represents

$$\frac{0.05}{0.336} = 0.15 \text{ of the motor torque.}$$

It should be noted that the available torque is expressed

$$T = K\theta$$

from which

$$\theta = \frac{T}{K}.$$

When the system is stationary, there is no error due to velocity. However, a torque of 5 ft.-lb. is produced by the wind load. An equal and opposite counteracting torque must be developed by the controller to hold the system stationary. The error which must be tolerated to produce this torque is equal to

$$\theta = \frac{T}{K} = \frac{5}{480} \times \frac{360}{2\pi} = 0.6 \text{ deg.}$$

Negative Damping.—The dimensionless equation (4.65) is an expression of the input-output error of the servo system as a function of time, when the system is subjected to a step input velocity function. The first term ($2c$) of the right-hand member of the equation is independent of time and represents the steady-state error. The second term contains the exponential time factor $\epsilon^{-c\omega_n t}$ and represents the transient component of the error.

In the preceding discussion, the damping ratio c, which is equal to the friction ratio F/F_c, has been always taken as positive. Therefore the exponential factor $\epsilon^{-c\omega_n t}$ decreases for increasing values of time, and the transient term with which this factor is associated is *positively damped* and dies out.

The damping ratio c is positive because both friction terms F and F_c are positive. With such positive values, it was shown in relation to Figs. 4.16, 4.17, and 4.18, that the speed-torque characteristic of the system is represented by a curve that has a negative slope, the speed decreasing for increasing values of the torque. It was shown also that a similar speed-torque characteristic curve is obtained when an induction motor is used that has a high-resistance rotor (see Fig. 4.23, curve 3). No external friction damper is then necessary, and the friction coefficient F is then equal to the slope $-\Delta T/\Delta S$ of the motor speed-torque characteristic curve.

If, however, a motor were used, for which the speed increases with the torque (see for example the lower part of curve 2, Fig. 4.23), with only a small amount of mechanical damping in the output member of the system, the factor c would become negative. The exponential factor $\epsilon^{-c\omega_n t}$, and therefore the transient term in Eq. (4.65), would then increase in magnitude with increasing values of the time t. The system is then unstable. If $0 > c > -1$, an oscillation builds up in the system, growing in amplitude until limited by a change in the slope of the speed-torque characteristic of the system, when a condition of steady *hunting* results. If $-1 > c$, the output member accelerates continuously in a same direction, and the error increases without oscillation, until limited as in the previous case, when the entire process collapses. The system is then subject to so-called *relaxation oscillations.*[1]

Similar instability conditions may arise when appreciable time lags are present in one of the signal channels of the system.

[1] Relaxation oscillations have been studied extensively, in particular by B. van der Pol, Relaxation Oscillations, *Phil Mag.*, vol. 2, 1926; Oscillations Sinusoidales et de Relaxation, *L'Onde Electrique,* June and July, 1930.

CHAPTER V

ANALYSIS OF SERVOMECHANISMS WITH ERROR-RATE DAMPING

Comparison between Viscous Damping and Error-rate Damping.—As in the preceding chapters, the servomechanisms to be discussed here establish a correspondence between the positions of the input and output members of the system. Whenever this correspondence is altered, the resulting discrepancy or error between the positions of the input and output members causes the controller to develop a force of such magnitude and direction as to tend to reduce the error to zero. In the systems previously discussed, this force is proportional to the error, and is thus equivalent to the elastic force of a spring, which tends to restore the spring to its original length, whenever this length is altered by a stretching or compression of the spring.

Since in addition to this elasticity the system is also endowed with inertia, it will oscillate when it is disturbed from its steady-state position of equilibrium. During such oscillation the output member no longer follows accurately the position variations of the input member. To reduce the amplitude and duration of the oscillation is the function of such damping as may be introduced in the system.

The form of damping considered in the preceding chapter, *viz.*, viscous damping, consists in applying to the system a retarding force proportional to the speed of the output member. While such viscous damping effectively reduces transient oscillations, it also produces an undesired steady-state error, which results from the very fact that the output member is moving. In other words, viscous damping not only reduces the oscillation, but also retards the output member of the system. In order to eliminate this effect, it is then necessary to resort to other forms of damping or stabilization, which will operate only on the oscillation or transient error, but not on the steady-state motion of the system.

The difference between the two types of damping can perhaps best be understood by describing a simple mechanical system that behaves exactly like the servomechanisms here discussed. However, it should be emphasized that such a mechanical system is *not* a servo, and that it is described only because its motion characteristics are similar to those of a servomechanism.

Before such a system is described, the simple pendulum shown in Fig. 5.1 may be considered. This consists of a mass M suspended from

a pivot O by a thin rod OM. When the pendulum is at rest, it hangs in a vertical direction, which is its position of stable equilibrium. If the mass M is temporarily displaced to some other position M' and then released, the pendulum oscillates back and forth between positions OM' and OM'' equally displaced on each side of the vertical position OM.

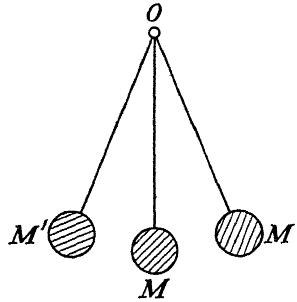

FIG. 5.1.—Simple pendulum.

In the absence of any friction the oscillation persists indefinitely with constant amplitude. On the other hand, if some friction is present, each swing of the mass M is smaller than the preceding one by a constant percentage, and the pendulum more or less rapidly comes to rest in its original vertical position OM; *i.e.*, the oscillation is damped.

Referring now to the system shown in Fig. 5.2, suppose that the pivot O of the pendulum is mounted on a carriage C that may be made to move horizontally along a rail RR'. As will be shown below, the behavior of such an arrangement can be compared to that of a servomechanism, the input and output member positions of which are represented, respectively, by the position of the carriage along the rail and by the position of the pendulum mass. The difference between the positions of the carriage

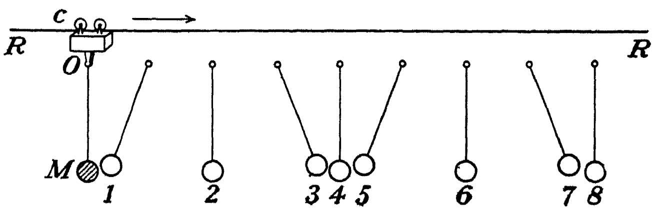

FIG. 5.2.—Pendulum analogy of undamped servomechanism.

and of the mass along the direction of motion at any instant corresponds to the error of the servomechanism. The response of the pendulum mass to motion of the carriage is, with comparable constants, identical with the response of a servo system to the motion of its input member.

Starting from the position of rest, shown at the left of Fig. 5.2, and for which the pendulum is in its stationary vertical position, let the carriage be *suddenly* set in motion at *constant* speed toward the right. The figure then shows the position of the pendulum and its support at equal intervals of time after the starting instant. By virtue of its inertia, the mass M at first lags behind the pivot O in its motion toward the right, and the pendulum assumes a slanting position shown at 1. As the carriage and pivot move on farther, the pendulum swings forward,

through positions 2 and 3, then backward (positions 4 and 5), oscillating back and forth indefinitely with constant amplitude.

Thus, the *average* position of the pendulum mass M follows the position of the carriage C in its constant-speed motion toward the right, while the *instantaneous* position of the mass alternately precedes and lags behind that of the carriage. This creates an alternately positive and negative discrepancy, or error, between the positions of the mass M on the one hand, and the pivot O and carriage C on the other. The system is thus comparable to an undamped servo control system, in which the carriage and pivot represent the input member, while the mass represents the output member, and the *zero-error* position correspondence is that for which the pivot and mass are on a common vertical line.

Similarly, the system shown in Fig. 5.3 may be taken as equivalent to a viscous-damped servo system. It differs from the system of Fig. 5.2 in that the mass M is provided with a small vane that dips in some

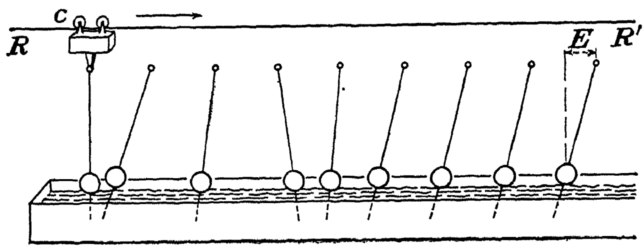

FIG. 5.3.—Pendulum analogy of viscous-damped servomechanism.

damping liquid, such as oil, for example, contained in a tank beneath the rail RR'. As the carriage is suddenly started from the position of rest shown at the left, the pendulum at first oscillates, as in the previous example. However, the retarding force between the vane and the liquid damps the oscillation more or less rapidly, and the pendulum mass M then simply follows the constant-speed motion of the carriage and pivot toward the right. But, under these conditions, the pendulum does *not* assume a vertical position. The friction arising between the vane and the liquid, because of the motion, produces a retarding force on the mass M as the vane is dragged through the liquid by the motion of the carriage. Thus, the position of the mass M lags behind that of the pivot O in the motion of the system toward the right, and the resulting *error* E between the instantaneous positions of the mass and pivot is proportional to the speed of motion. A force equal and opposite to the retarding friction force must be constantly applied to the carriage in order to keep the system moving at constant speed. Under steady-state conditions the pendulum assumes a slanting position of equilibrium for which the torques due to friction and gravity are equal and opposite.

From the preceding remarks, it is seen that the mass M, which represents the output member of a servo system as studied before, is subjected to two kinds of motion superimposed upon each other and constituting its actual motion as just described. One of these two component motions is a constant-speed translation toward the right. The other is a temporary or transient damped oscillatory motion, lasting only during and directly following the accelerating period of the system. Both these motions are affected by the friction, which is proportional to the speed of the mass M with respect to the damping liquid in the tank.

Since the steady-state error is due to the constant-speed translatory component of the motion of the mass and its vane with respect to the liquid, it becomes obvious that no such error will arise if this relative translatory motion between the mass and the damping liquid can be eliminated. At the same time, the system must provide for the damping

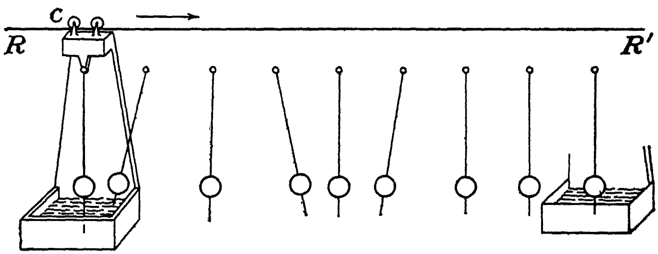

FIG. 5.4.—Pendulum analogy of error-rate-damped servomechanism.

of oscillations due to acceleration (or deceleration) of the input member, or carriage C.

Such a system is illustrated in Fig. 5.4. Instead of using a stationary tank to contain the damping liquid, as was done in the case of Fig. 5.3, the present arrangement employs a smaller tank permanently attached to the carriage C, or input member of the system. When the carriage is suddenly started from rest and set in constant-speed motion toward the right, the pendulum at first oscillates in the same manner as described before. The friction arising from the resulting motion of the vane in the liquid soon damps out this oscillation. But since the motion of the carriage C does not drag the pendulum mass through the damping liquid (which it did in the case of Fig. 5.3), the pendulum reverts to a *vertical* position of rest. In other words, while an oscillation of the pendulum produces a relative motion of the mass and the liquid, which motion is then damped out, there is *no* relative translatory motion and therefore *no steady-state error*. The carriage, pendulum and damper all move together at the same constant speed toward the right, and the mass M and pivot O retain their original relative positions on a common vertical line.

In order to apply to the design of a servomechanism the principles

involved in such an arrangement, it is necessary to analyze further the particular type of damping illustrated here. This damping affects only the oscillatory component of motion, without operating on the constant-speed translatory motion component. It will be recalled that the friction force, which produces the damping, is proportional to the *speed* of the mass and its vane with respect to the damping liquid. Since the damping liquid is carried along by the carriage in its uniform translatory motion, the damping force is proportional to the speed of the mass with respect to the carriage, which is actually the speed of the error. Thus, in the system of Fig. 5.4, damping is proportional to the *rate of change of the error*, instead of being proportional to the output speed as in the case of the systems studied in the previous chapters.

In a servo system employing such error-rate damping, the controller must therefore produce a driving force or torque proportional to the sum of the following two components.

1. The input-output position error, since this is the force through which the output member is made to follow the input member when the input member position is changed, *i.e.*, when the input member is moved with respect to the output member. (Note that in the preceding pendulum analogies the output position restoring force is the force of gravity, which tends to bring the pendulum back to the vertical position. For small oscillation angles of the pendulum this force is proportional to the displacement of the mass M from the vertical, *i.e.*, the force is proportional to the position error between the pendulum mass and pivot.)

2. The rate of change of the input-output position error. This component of the controller torque has been shown, in the pendulum analogy, to provide damping without introducing a steady-state error.

This is expressed mathematically by the relation

$$T = K\theta + L\frac{d\theta}{dt} \tag{5.1}$$

where T is the torque produced by the controller, θ and $d\theta/dt$ are the input-output error and time rate of change of the error, respectively, and the proportionality factors K and L are two controller constants.

Examples of Error-rate-damped Servomechanisms.—Before proceeding with a mathematical analysis of the operating properties and design requirements of error-rate-damped servomechanisms, a few elementary examples of such mechanisms are briefly described here, for the purpose of giving a better understanding of the process involved. The basic difference between these devices and those studied in the previous chapters was shown to be that the signal which actuates the controller must indicate the magnitude and direction of the input-output position error, as well as the time rate of change of the error.

This object can be accomplished by mechanical means, such as a gyroscopic device, for instance, in which use is made of its precessional motion. Electromechanical devices, as described below, and certain electrical networks to be described in Chap. VII, can also be used. An illustration of an error-rate-damped servomechanism is shown schematically in Fig. 5.5. Like previously described servos, it comprises an input member, an output member with connected load, and a differential device that indicates the input-output error by the angular position of its *error shaft*. Mechanically coupled to this shaft is the slider of a potentiometer P, the resistance element of which is connected across a battery B. Also driven by the differential error shaft is a d-c generator G. One terminal of this generator is connected to the potentiometer slider.

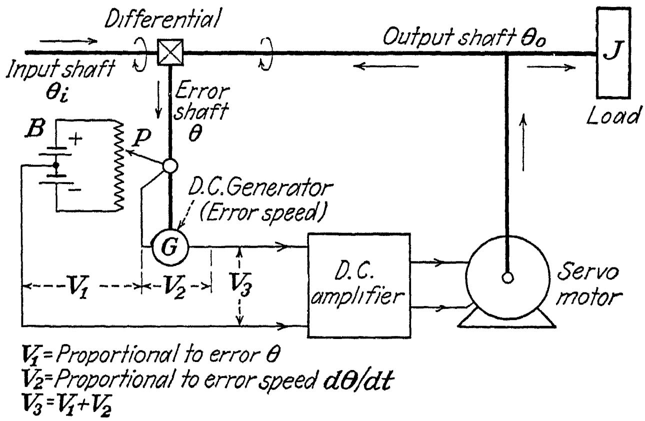

FIG. 5.5.—Error-rate-damped servomechanism.

The other generator terminal and the center tap of the battery are connected to the input terminals of a d-c amplifier, the output terminals of which are connected to the servo motor of the system. This servo motor, either directly or through a suitable gear, drives the output shaft and load.

Under these conditions, a constant input-output error causes the error shaft of the differential device to be angularly displaced from its *zero-error* position, by an amount that is equal or proportional to the error. The potentiometer slider P, which is locked on the error shaft, is shifted by a corresponding amount, and the voltage V_1 across the potentiometer, therefore, indicates the magnitude and direction of the error.

If the error varies to some other value, the angular position of the error shaft and potentiometer slider changes accordingly. During the error variation, the error shaft at the same time rotates the armature of the generator G by a certain angle and causes the generator to develop,

during the period of the change, a voltage V_2 that is proportional to the speed of the change. This speed is equal to the *time rate of change of the error.*

Thus, the voltage V_3 that is impressed on the input terminals of the amplifier feeding the servo motor is the algebraic sum of the voltages V_1 and V_2 representing, respectively, the error and the error rate of the servomechanism.

Fundamental Parameters of Error-rate-damped Servomechanisms.—In the preceding paragraphs servo systems were described which are stabilized through error-rate damping instead of viscous output damping. A servomechanism with error-rate damping is represented schematically in Fig. 5.6. As in the case of viscous output damped systems, it comprises an input member, an output member connected to a load that has a moment of inertia J, a differential device to indicate the input-output position error, and a controller that is actuated from the differential

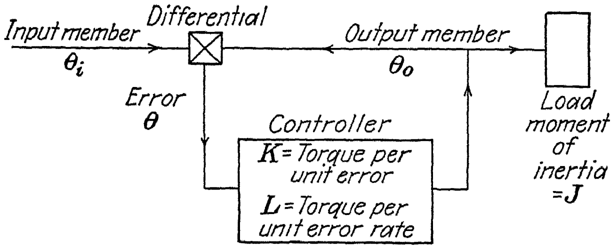

FIG. 5.6.—Error-rate-damped servomechanism.

device and develops a torque for driving the load. This system differs from a viscous-damped servo in that the output member is not subjected to a retarding force or friction proportional to the output speed, while the controller output torque is proportional to the time rate of change of the error as well as to the error. The error-rate feature requires devices to be associated with the system for indicating both the error and its time rate of change.

In the following analysis of such a system, the same symbols will be used as in the preceding chapter. With no viscous damping in the output member, the factor F will be equal to zero. On the other hand, a factor L is introduced, to represent the constant of proportionality between the controller output torque and the time rate of change of the error.

Symbol	Quantity	C.g.s. unit	F.p.s. unit
L	Torque per unit error rate	$\dfrac{\text{Dyne-cm.}}{\text{Radians per sec.}}$	$\dfrac{\text{Ft-lb.}}{\text{Radians per sec.}}$

Equation of the System.—As before, the operating conditions of the system just described are expressed by stating that the sum of the forces acting on the system is equal to zero. In other words, the accelerating forces must equal the retarding forces. In the present instance, the accelerating force is the torque produced by the controller as a result of both the error and error rate of change. This must equal the retarding inertia force of the output member and load, which arises whenever the system is accelerated or decelerated.

$$K\theta + L\frac{d\theta}{dt} = J\frac{d^2\theta_o}{dt^2}. \tag{5.2}$$

Since the error θ is the difference between the input and output positions θ_i and θ_o,

$$\theta = \theta_i - \theta_o, \tag{5.3}$$

Eq. (5.1) can be rewritten in terms of input angle and error only by substituting in Eq. (5.1) the value of θ_o obtained from Eq. (5.3),

$$J\frac{d^2\theta}{dt^2} + L\frac{d\theta}{dt} + K\theta = J\frac{d^2\theta_i}{dt^2}. \tag{5.4}$$

Step Input Function.—As in the case of viscous-damped servomechanisms, the behavior of the present system will be investigated under the condition that the input member is motionless until some instant of time $t = 0$ when it is suddenly set in motion at constant angular velocity ω_1, the input angle θ_i thereby increasing linearly with time. This was shown graphically in Fig. 4.2 and is expressed

$$\theta_i = 0, \qquad t < 0, \tag{5.5a}$$
$$\theta_i = \omega_1 t, \qquad t \geq 0. \tag{5.5b}$$

For an input function of this type, the behavior of the system after the input member has been set in motion ($t > 0$) is expressed by a differential equation obtained by substituting in Eq. (5.4) the actual value of θ_i as given in Eq. (5.5*b*).

Successive differentiation of Eq. (5.5*b*) shows that

$$\frac{d^2\theta_i}{dt^2} = 0, \qquad t > 0. \tag{5.6}$$

Substituting this value in Eq. (5.4) gives the equation

$$J\frac{d^2\theta}{dt^2} + L\frac{d\theta}{dt} + K\theta = 0, \qquad t > 0, \tag{5.7}$$

the solution of which will express the input-output error θ of the system as a function of time.

Steady-state Error.—Equation (5.7) shows that after a sufficient time has elapsed for any transient to die out, and therefore for the error variations $d^2\theta/dt^2$ and $d\theta/dt$ to become equal to zero, the error θ itself must also vanish in order to make the term $K\theta$ equal to zero. Thus in a stable system with error-rate damping and no viscous output damping an input step function produces no *steady-state* error.

No steady-state error exists in such a system because, after the output load has been accelerated to the input speed, there is no output retarding force acting on the system, since the latter is assumed to be frictionless. No controller torque is then necessary to maintain the output load moving at the input speed, and consequently this steady-state speed can be sustained without any error signal being fed into the controller. This contrasts sharply with the case of a viscous-damped servomechanism, where a retarding torque, proportional to the output speed, prevents the output member from maintaining its speed in the absence of a driving torque from the controller. Comparison of Eq. (5.7) with the corresponding Eq. (4.8) for the viscous-damped case shows an additional term $F\omega_1$ in the latter, which gave rise to a steady-state lag.

Solution of the Equation.—Since there is no steady-state error, the complete solution of Eq. (5.7) reduces to a transient solution. The general transient solution will first be found, following a procedure similar to that used in the preceding chapter. The arbitrary constants that appear in this solution can then be determined from the boundary conditions of the system.

In comparing Eq. (5.7) with the homogeneous form of the differential equation of the error of a viscous-damped servo system Eq. (4.10), it is apparent that the equations are identical, except for different constant coefficients in the second terms. Therefore, it is reasonable to assume that the solution of Eq. (5.7) can be obtained by writing the generalized transient solution for the viscous-damped servo Eq. (4.32), and its auxiliary equations (4.21 and 4.22) with the error-rate constant L substituted for the output rate constant F

$$\theta = e^{-at}(B_1 \cos bt + B_2 \sin bt) \tag{5.8}$$

$$a = \frac{L}{2J} \tag{5.9}$$

$$b = \sqrt{\frac{K}{J} - \frac{L^2}{4J^2}}. \tag{5.10}$$

Two boundary conditions are required for determining the two constants B_1 and B_2. These conditions are the same as those previously determined for evaluating the constants of a viscous-damped servomechanism. First, from Eq. (4.34), the error is zero up to the instant

when the input member is set in motion.

$$\theta = 0, \qquad t \leq 0. \tag{5.11}$$

Second, from Eqs. (4.35) through (4.38), the error rate of change equals the input velocity at the instant the input member starts to move.

$$\frac{d\theta}{dt} = \omega_1, \qquad t = 0. \tag{5.12}$$

Applying the condition, Eq. (5.11), to Eq. (5.8),

$$B_1 = 0. \tag{5.13}$$

For applying the condition Eq. (5.12), it is necessary to differentiate Eq. (5.8).

$$\frac{d\theta}{dt} = \epsilon^{-at}(-B_1 b \sin bt + B_2 b \cos bt - B_1 a \cos bt - B_2 a \sin bt), \tag{5.14}$$

and at the time origin $t = 0$,

$$\frac{d\theta}{dt} = B_2 b - B_1 a. \tag{5.15}$$

Substituting Eqs. (5.12) and (5.13) in Eq. (5.15),

$$\omega_1 = B_2 b \tag{5.16}$$

from which

$$B_2 = \frac{\omega_1}{b}. \tag{5.17}$$

Placing in Eq. (5.8) the values of the constants B_1 and B_2, as found in Eqs. (5.13) and (5.17),

$$\theta = \frac{\omega_1}{b} \epsilon^{-at} \sin bt. \tag{5.18}$$

This expression is the solution of the problem. It gives the error angle θ as a function of the input speed ω_1 and system parameters a and b, as defined in Eqs. (5.9) and (5.10). Various operating characteristics will result, depending on the relative values of these parameters, as discussed in the following paragraphs.

Undamped System.—In the limiting condition where the damping factor L is zero, Eqs. (5.9) and (5.10) become

$$a = 0 \tag{5.19}$$

$$b = \sqrt{\frac{K}{J}}. \tag{5.20}$$

Substituting these expressions in Eq. (5.18), this becomes

$$\theta = \frac{\omega_1}{\sqrt{\frac{K}{J}}} \sin \sqrt{\frac{K}{J}}\, t. \tag{5.21}$$

This equation is exactly the same as Eq. (4.49), which expresses the error time function of a viscous-damped servomechanism which has a damping constant F equal to zero. As noted in this previous case, the error initiated in such a system by a step input function is a sinusoidal time function of constant amplitude. This undamped oscillation, which continues indefinitely, has a radian frequency defined as the natural frequency of the system. This frequency is given in Eq. (4.50), which is rewritten here.

$$\omega_n = \sqrt{\frac{K}{J}}. \tag{5.22}$$

Critically Damped System.—When the value of the term $L^2/4J^2$ in Eq. (5.10) tends toward that of the term K/J, the parameter b approaches zero. Rewriting Eq. (5.18) in the form

$$\theta = \omega_1 \epsilon^{at} \frac{\sin bt}{b}, \tag{5.18a}$$

the term $(\sin bt)/b$ tends toward t.

$$\frac{L^2}{4J^2} \rightarrow \frac{K}{J} \tag{5.23}$$

$$b \rightarrow 0$$

$$\frac{\sin bt}{b} \rightarrow t. \tag{5.24}$$

Substituting Eq. (5.24) in Eq. (5.18), the error is then expressed

$$\theta = \omega_1 t \epsilon^{-at}. \tag{5.25}$$

In this limiting case known as *critical damping*, the error thus tends toward zero without oscillation. From Eq. (5.23), the amount of error-rate damping L_c necessary to damp the system critically is seen to be

$$L_c = 2\sqrt{KJ}. \tag{5.26}$$

Underdamped System.—For systems that are not critically damped, a parameter c known as the *damping ratio* is used, which is similar to that previously defined in Eq. (4.57). This parameter c is the ratio of the actual damping to the damping necessary to damp the system critically.

$$c \equiv \frac{L}{L_c}. \tag{5.27}$$

Combining the Eqs. (5.26) and (5.27), there results

$$L = 2c\sqrt{KJ}. \tag{5.28}$$

Substitution of Eqs. (5.22) and (5.28) in Eqs. (5.9) and (5.10) gives, as in the viscous-damped servo,

$$a = c\omega_n \tag{5.29}$$
$$b = \omega_n\sqrt{1 - c^2}. \tag{5.30}$$

Dimensionless Form of Equations.—Inserting Eq. (5.29) in the *critically damped* equation (5.25), for which $c = 1$, the equation becomes

$$\theta = \omega_1 t\epsilon^{-\omega_n t}, \tag{5.31}$$

or, in dimensionless form

$$\theta\frac{\omega_n}{\omega_1} = \omega_n t\epsilon^{-\omega_n t}. \tag{5.32}$$

Similarly, the *general oscillatory* equation (5.18) is written in terms of ω_n and c by use of Eq. (5.29) and (5.30).

$$\theta = \frac{\omega_1}{\omega_n\sqrt{1 - c^2}}\epsilon^{-c\omega_n t}\sin\omega_n\sqrt{1 - c^2}\,t, \tag{5.33}$$

or, in dimensionless form

$$\theta\frac{\omega_n}{\omega_1} = \frac{1}{\sqrt{1 - c^2}}\epsilon^{-c\omega_n t}\sin\omega_n\sqrt{1 - c^2}\,t. \tag{5.34}$$

Equations (5.32) and (5.34) express in dimensionless form the error of an error-rate-damped servomechanism with a suddenly applied input velocity.

These equations apply for values of the damping ratio c between zero and unity. In the case where $c > 1$, which occurs when the servo system is overdamped, Eq. (5.34) may be rewritten in the form

$$\theta\frac{\omega_n}{\omega_1} = \frac{1}{\sqrt{c^2 - 1}}\epsilon^{-c\omega_n t}\sinh\omega_n\sqrt{c^2 - 1}\,t. \tag{5.34a}$$

Discussion and Use of Dimensionless Equations.—The relations, Eqs. (5.32), (5.34), and (5.34a), just found for the critically damped, oscillatory, and overdamped cases, respectively, can be represented graphically to show the error $\theta(\omega_n/\omega_1)$ as a function of time $\omega_n t$, for various values of the damping ratio $c \doteq L/L_c$ of the system. Such curves are plotted in the graph of Fig. 5.7, from which it is seen that, for positive values of the damping ratio, the error tends asymptotically toward zero.

The time required for the transient error to become vanishingly small is a minimum when the system is critically damped ($c = 1$). When the system is underdamped ($c < 1$), the error is a damped oscillatory function of time. The smaller the value of the damping ratio, the greater is the initial error swing.

When applying Eqs. (5.32), (5.34), and (5.34*a*), or when using the curves of Fig. 5.7, the units employed for expressing the error θ, natural frequency ω_n, input speed ω_1, and time t must be chosen in such manner that the factors $\theta\,(\omega_n/\omega_1)$ and $\omega_n t$ shall be dimensionless. This was explained in Chap. IV in relation to the use of dimensionless equations.

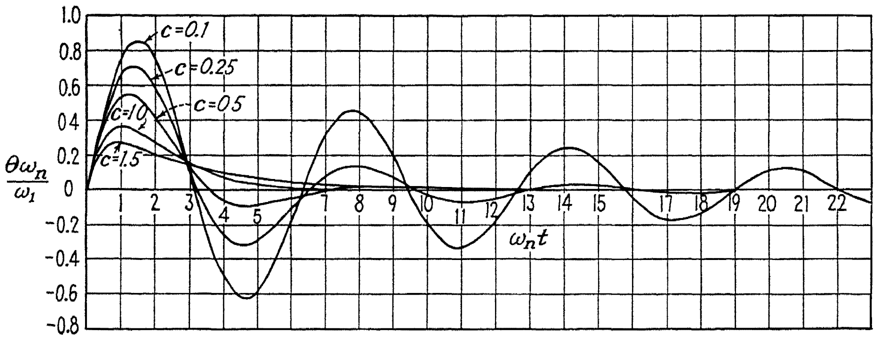

FIG. 5.7.—Dimensionless error-time curves for error-rate-damped servomechanism subjected to a step input velocity function.

Torque Variations as Functions of Time.—The fact that the steady-state error tends toward zero irrespective of the value of the damping ratio is not the only feature that distinguishes the operation of error-rate-damped systems from that of the viscous-damped systems discussed in the preceding chapter. In a viscous-damped system the torque developed by the controller is proportional to the error and is expressed $K\theta$. The curves which represent torque as a function of time are then of the same shape as those which represent the error as a function of time. This is not the case for an error-rate-damped servo system, since the torque is then proportional to both the error and time rate of change of the error, as expressed by the relation

$$T = K\theta + L\frac{d\theta}{dt}. \tag{5.1}$$

Recalling Eqs. (5.28) and (5.22),

$$L = 2c\sqrt{KJ} \tag{5.28}$$

$$\omega_n = \sqrt{\frac{K}{J}} \tag{5.22}$$

and combining these,

$$\frac{L}{K} = 2c\sqrt{\frac{J}{K}} = \frac{2c}{\omega_n}, \tag{5.35}$$

the torque may be expressed

$$T = K\left(\theta + \frac{2c}{\omega_n}\frac{d\theta}{dt}\right). \tag{5.36}$$

Consider now a critically damped error-rate system, in which the error θ is expressed, as a function of time, by Eq. (5.31).

$$\theta = \omega_1 t\epsilon^{-\omega_n t}. \tag{5.31}$$

Differentiating this equation,

$$\frac{d\theta}{dt} = \omega_1(1 - \omega_n t)\epsilon^{-\omega_n t}, \tag{5.31a}$$

and writing these values of θ and $d\theta/dt$ in the above torque relation, Eq. (5.36), in which c is set as equal to unity for critical damping, the torque is expressed as a function of time by the equation

$$T = K\frac{\omega_1}{\omega_n}[\omega_n t\epsilon^{-\omega_n t} + 2(1 - \omega_n t)\epsilon^{-\omega_n t}]. \tag{5.37}$$

In this expression, the first term represents the torque component due to the error, while the second term represents the torque component due

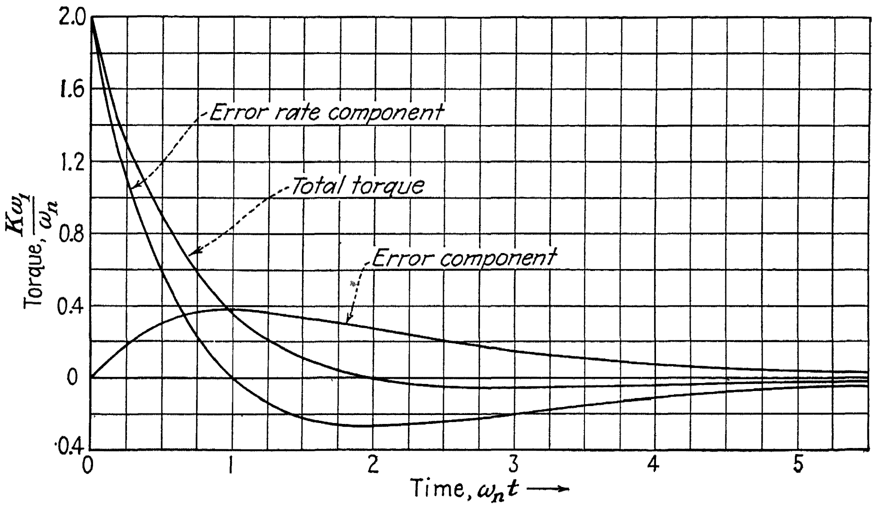

FIG. 5.8.—Torque variations as functions of time in an error-rate-damped servomechanism with critical damping.

to the error rate of change. These two component torques, as well as the total resultant torque, are plotted as functions of $\omega_n t$ in the graph of Fig. 5.8 in dimensionless form. At the time origin, the error component

of the torque is equal to zero, while the error-rate torque component is a maximum. This is because the instant after the input member of the system is suddenly started (the output member being previously at rest) the error rate equals the velocity of the input member, while the error is still approximately zero.

After the output member has accelerated sufficiently, the error and the error component of the torque decrease. While the error is decreasing, the error rate and the error-rate torque component become negative. It should be noted that at the time $\omega_n t = 1$, the error and the error torque are maximum while the error rate and the error-rate torque are zero. Also, at the time $\omega_n t = 2$, the total torque is zero, while both error and error-rate torque components are equal and opposite.

In other words, and as illustrated by the curves of Fig. 5.8, the error component of the torque is always positive and tends to accelerate the

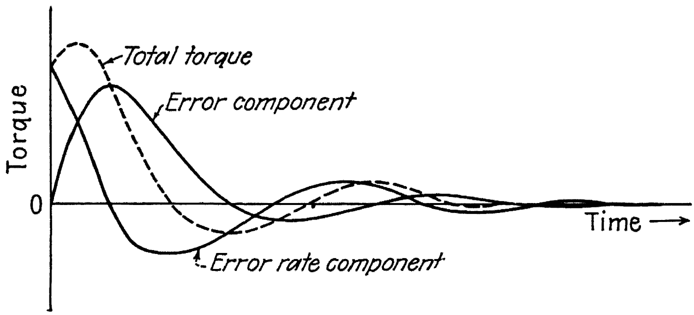

FIG. 5.9.—Torque variations as functions of time in an error-rate-damped servomechanism with less-than-critical damping.

load. On the other hand, the error-rate component of the torque is at first positive, and by adding to the error component of the torque, helps to accelerate the load rapidly from its condition of rest. Then, as the load accelerates, the error rate component of the torque becomes negative, and by *holding back* the load, prevents it from overshooting its final correspondence position with respect to the input member.

Similar curves may be plotted for an underdamped servo system ($c < 1$), by writing the values of θ and $d\theta/dt$ obtained from Eq. (5.33) in Eq. (5.1). This is shown in Fig. 5.9 for a damping ratio $c = 0.25$.

Response to Sinusoidal Input Function.—The procedure used to find the steady-state frequency response of an error-rate-damped servomechanism is the same as that used to find the frequency response of a viscous-damped servo. First, the differential equation of the output member position in terms of the input member position is determined. A sinusoidal input function is used, and it is assumed that the output function is of similar form, differing only in amplitude and phase. The amplitude and phase characteristics are then determined by substitution

of the input and output functions in the servo equation. Finally, the amplitude and phase equations are rewritten in dimensionless form with the aid of convenient parameter symbols.

The equation of motion of an error-rate-damped servomechanism was previously given in Eq. (5.2).

$$K\theta + L\frac{d\theta}{dt} = J\frac{d^2\theta_o}{dt^2}, \tag{5.2}$$

where

$$\theta = \theta_i - \theta_o. \tag{5.3}$$

The differential equation relating the input and output positions and displacements is obtained by substituting in Eq. (5.2) the equivalent of θ, as given in Eq. (5.3).

$$K\theta_i + L\frac{d\theta_i}{dt} = J\frac{d^2\theta_o}{dt^2} + L\frac{d\theta_o}{dt} + K\theta_o. \tag{5.38}$$

If a linear equation such as Eq. (5.38) is subjected to a unit complex exponential function

$$\theta_i = \epsilon^{j\omega t}, \tag{5.39}$$

the output function will be of the form

$$\theta_o = A\epsilon^{j(\omega t+\lambda)}. \tag{5.40}$$

The values of A and λ obtained by the use of these functions will apply also for the case where the input function is simply a sinusoidal function of unit amplitude

$$\theta_i = \cos \omega t, \tag{5.41}$$

the output function being then

$$\theta_o = A \cos (\omega t + \lambda). \tag{5.42}$$

The validity of this transformation was discussed in the preceding chapter.

The first time derivative of the input function, Eq. (5.39), is

$$\frac{d\theta_i}{dt} = j\omega\epsilon^{j\omega t}. \tag{5.43}$$

The first and second time derivatives of the output function, Eq. (5.40), are

$$\frac{d\theta_o}{dt} = j\omega A\epsilon^{j(\omega t+\lambda)} \tag{5.44}$$

$$\frac{d^2\theta_o}{dt^2} = -\omega^2 A\epsilon^{j(\omega t+\lambda)}. \tag{5.45}$$

Substituting Eqs. (5.39), (5.40), (5.43), (5.44), and (5.45) in Eq. (5.38) gives

$$K\epsilon^{j\omega t} + j\omega L\epsilon^{j\omega t} = -\omega^2 JA\epsilon^{j(\omega t+\lambda)} + j\omega LA\epsilon^{j(\omega t+\lambda)} + KA\epsilon^{j(\omega t+\lambda)}. \quad (5.46)$$

Dividing through by $\epsilon^{j\omega t}$, this equation becomes

$$K + j\omega L = -\omega^2 JA\epsilon^{j\lambda} + j\omega LA\epsilon^{j\lambda} + KA\epsilon^{j\lambda}, \quad (5.47)$$

or

$$A\epsilon^{j\lambda} = \frac{K + j\omega L}{K - \omega^2 J + j\omega L}. \quad (5.48)$$

Dividing the numerator and denominator of the right-hand member by K, this expression becomes

$$A\epsilon^{j\lambda} = \frac{1 + j\omega\frac{L}{K}}{1 - \omega^2\frac{J}{K} + j\omega\frac{L}{K}}. \quad (5.49)$$

Substituting the values of J/K and L/K, as obtained from Eq. (5.22) and (5.35),

$$\frac{J}{K} = \frac{1}{\omega_n^2} \quad \text{and} \quad \frac{L}{K} = \frac{2c}{\omega_n},$$

Eq. (5.49) is written

$$A\epsilon^{j\lambda} = \frac{1 + j2c\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2} + j2c\frac{\omega}{\omega_n}}. \quad (5.50)$$

Introducing, as in the preceding chapter, a variable d to denote the relative operating frequency,

$$d \equiv \frac{\omega}{\omega_n}, \quad (5.51)$$

and substituting Eq. (5.51) in Eq. (5.50),

$$A\epsilon^{j\lambda} = \frac{1 + j2cd}{1 - d^2 + j2cd}. \quad (5.52)$$

Equation (5.50), or its equivalent form Eq. (5.52), may be considered as representing a vector of magnitude A and phase angle λ (both relative to the input displacement taken as a unit reference vector) equal, respectively, to

$$\left\{\begin{aligned} A &= \sqrt{\frac{1 + 4c^2 \dfrac{\omega^2}{\omega_n^2}}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4c^2 \dfrac{\omega^2}{\omega_n^2}}} = \sqrt{\frac{1 + 4c^2d^2}{(1 - d^2)^2 + 4c^2d^2}} \qquad (5.53) \\ \lambda &= \tan^{-1} 2c \frac{\omega}{\omega_n} - \tan^{-1}\left(\frac{2c \dfrac{\omega}{\omega_n}}{1 - \dfrac{\omega^2}{\omega_n^2}}\right) = \tan^{-1} 2cd - \tan^{-1} \frac{2cd}{1 - d^2}. \qquad (5.54) \end{aligned}\right.$$

For an input function of the form given in Eq. (5.41)

$$\theta_i = \cos \omega t, \tag{5.41}$$

the output function of the servo system is then found by substituting Eqs. (5.53) and (5.54) in Eq. (5.42),

$$\theta_o = \sqrt{\frac{1 + 4c^2d^2}{(1 - d^2)^2 + 4c^2d^2}} \cos\left(\omega t + \tan^{-1} 2cd - \tan^{-1} \frac{2cd}{1 - d^2}\right). \tag{5.55}$$

Resonance Curves.—Equation (5.55) expresses, as a function of time, the instantaneous position θ_o of the output member of an error-rate servo system, when the input member is displaced back and forth according to a sinusoidal time function Eq. (5.41). Both functions have the same frequency.

The relative amplitude A of the output member oscillation is expressed in Eq. (5.53) as a function of the damping ratio c and the ratio d of the input frequency ω to the natural frequency ω_n of the system. For a servo system of given damping ratio c, this amplitude is therefore a function of d and hence of the input frequency. This is shown in Fig. 5.10, in which several resonance curves have been plotted, each curve corresponding to some particular value of the damping ratio c.

The value of d for which the output oscillation amplitude A is a maximum may be calculated by differentiating Eq. (5.53) with respect to d, and equating to zero. This value is

$$d_{A\,\max} = \frac{1}{2c} \sqrt{\sqrt{8c^2 + 1} - 1} \tag{5.56}$$

for which the oscillation amplitude is

$$A_{\max} = \sqrt{\frac{8c^4}{8c^4 - 4c^2 - 1 + \sqrt{8c^2 + 1}}}. \tag{5.57}$$

It is interesting to compare the curves of Fig. 5.10 with the corresponding curves of viscous-damped servo systems, as given in Fig. 4.11 of the preceding chapter. It was found there that the resonance curve has a maximum only for such systems in which the damping ratio c is smaller

than $1/\sqrt{2}$. No such limitation exists in the present case of error-rate-damped servo systems, since the value of d, as given in Eq. (5.56), is always real, irrespective of the value of c. Thus, the resonance curve always has a maximum.

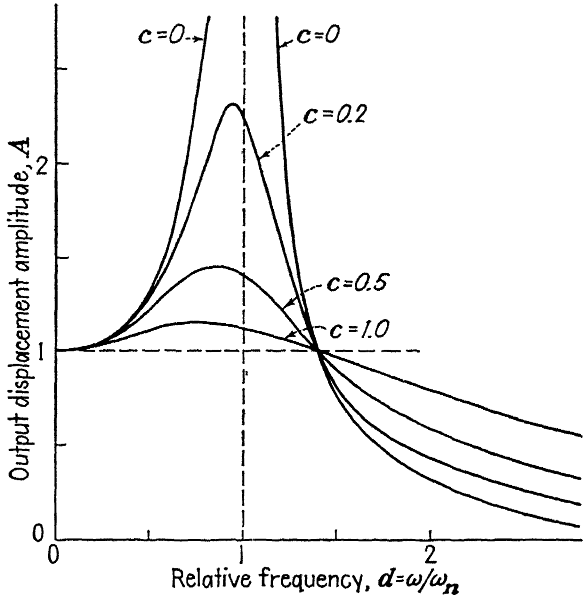

FIG. 5.10.—Resonance curves of servomechanism with error-rate damping for various values of damping ratio.

On the other hand, the resonance curves of Fig. 5.10 all cross each other at a point corresponding to a value of d equal to

$$d = \frac{\omega}{\omega_n} = \sqrt{2} \tag{5.58}$$

for which the amplitude A is equal to

$$A = 1, \tag{5.59}$$

irrespective of the value of the damping ratio c.

This property affords a simple means for experimentally measuring the natural frequency ω_n of the servo system, without regard to its amount of damping: it is only necessary to displace the input member back and forth sinusoidally with time at such a frequency ω_x that the output and input amplitudes are equal. The natural frequency ω_n of the system is then equal to

$$\omega_n = \frac{\omega_x}{\sqrt{2}}. \tag{5.60}$$

CHAPTER VI

ANALYSIS OF SERVOMECHANISMS WITH COMBINED VISCOUS OUTPUT DAMPING AND ERROR-RATE DAMPING

In the purely error-rate-damped servo systems studied in the preceding chapter, torque was shown to be developed by the controller only during periods when the speed is varied from some steady value to some other steady value. The torque was then found to be proportional to the input-output position error and to the time rate of change of this error.

These conditions imply that there is no viscous damping in the system, or in other words, that there is no torque component dependent on

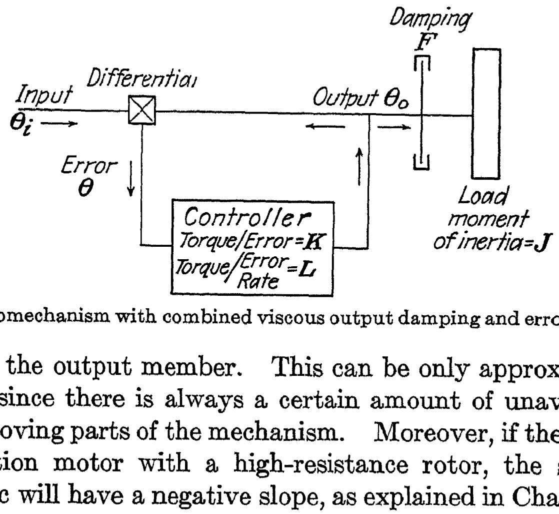

FIG. 6.1.—Servomechanism with combined viscous output damping and error-rate damping.

the speed of the output member. This can be only approximately true in practice, since there is always a certain amount of unavoidable friction in the moving parts of the mechanism. Moreover, if the servo motor is an induction motor with a high-resistance rotor, the speed-torque characteristic will have a negative slope, as explained in Chap. IV. This is equivalent to a viscous friction retarding torque, since the torque then decreases as the speed of motion increases.

It thus becomes necessary to consider the operating conditions of a servo control system having combined viscous output damping and error-rate damping. Such a system is represented schematically in Fig. 6.1. As in the previous cases, the servo system has an inertia load connected to its output member. The angular positions of the output and input members are compared by a differential device, and their difference, or error, actuates a controller. The damping provided in the system is

similar to that of a viscous-damped servomechanism in that the output member is subjected to a retarding force (or friction) proportional to the output speed. The damping also resembles that of an error-rate-damped servo system, since the controller is made to produce an output torque proportional to both the error and the rate of change of the error.

Equation of the System.—As in other servomechanisms, the algebraic sum of the forces acting on the system is equal to zero. In other words, the accelerating forces must be equal to the retarding forces. In the present case, the accelerating force is the torque produced by the controller and comprises two components:

$K\theta$, the torque proportional to the input-output position error.

$L\dfrac{d\theta}{dt}$, the torque proportional to the rate of change of the error.

The retarding force also has two components:

$J\dfrac{d^2\theta_o}{dt^2}$, the retarding torque of the output inertia, proportional to the output acceleration.

$F\dfrac{d\theta_o}{dt}$, the viscous drag due to the output friction, proportional to the output speed.

The equation may thus be written

$$K\theta + L\frac{d\theta}{dt} = J\frac{d^2\theta_o}{dt^2} + \frac{F\,d\theta_o}{dt}. \tag{6.1}$$

Since the error θ is the difference between the input and output angular positions,

$$\theta = \theta_i - \theta_o, \tag{6.2}$$

Eq. (6.1) can be rewritten in terms of input angle and error only.

$$J\frac{d^2\theta}{dt^2} + (L + F)\frac{d\theta}{dt} + K\theta = J\frac{d^2\theta_i}{dt^2} + F\frac{d\theta_i}{dt}. \tag{6.3}$$

This equation should be compared with Eq. (4.3) and (5.4) for the cases of pure viscous output damping and pure error-rate damping, respectively.

Step Input Function.—As in the cases considered before, a study will be made of the behavior of the system when this is subjected to a step input velocity function. The system is motionless until some instant $t = 0$, and its input member is then suddenly set in motion at constant speed ω_1, causing the input angle to increase linearly with time.

$$\theta_i = 0, \qquad t < 0. \tag{6.4}$$
$$\theta_i = \omega_1 t, \qquad t \geqq 0. \tag{6.5}$$

This condition is represented graphically in Fig. 4.2.

The differential equation of the motion of the system after the starting instant, *i.e.*, for $t > 0$, is found by substituting in Eq. (6.3) the successive time derivatives of θ_i.

$$\frac{d\theta_i}{dt} = \omega_1, \qquad t > 0 \tag{6.6}$$

$$\frac{d^2\theta_i}{dt^2} = 0, \qquad t > 0. \tag{6.7}$$

Thus, Eq. (6.3) becomes

$$J\frac{d^2\theta}{dt^2} + (L + F)\frac{d\theta}{dt} + K\theta = F\omega_1, \qquad t > 0, \tag{6.8}$$

which is the equation of motion of the error.

A solution of this equation will express the error θ as a function of time. As before, the steady-state error will first be found. Next, the form of the transient, through which the system reaches its steady-state operating condition, will be determined. Finally, the steady-state and transient solutions will be combined, giving the complete solution of the equation. The arbitrary constants that appear in this solution can then be found from the boundary conditions of the servo system.

Steady-state Error.—For the type of input function applied here, the steady-state error is constant; consequently its first and second time derivatives are equal to zero. Hence, from Eq. (6.8), the steady-state error is

$$\theta_S = \frac{F\omega_1}{K}. \tag{6.9}$$

It will be noted that this steady-state error is a function of the input speed ω_1, controller constant K, and viscous output damping coefficient F, but *not* of the error-rate damping coefficient L.

Transient Error.—The generalized solution of the transient error is dependent only on the characteristics of the servo system itself, and is not affected by the form of the input function. Consequently, the input function, which appears in the second member of Eq. (6.3), can for convenience be set equal to zero, and the equation becomes

$$J\frac{d^2\theta}{dt^2} + (L + F)\frac{d\theta}{dt} + K\theta = 0. \tag{6.10}$$

In comparing this Eq. (6.10) with the homogeneous form of the differential error equation of a viscous-damped system, Eq. (4.10), it is seen that the equations are identical, except for the values of the constant coefficients of their second terms. It is, therefore, permissible to use for the solution of Eq. (6.10) the same generalized transient solution as was

found in Eq. (4.32) and the auxiliary equations (4.21) and (4.22) for the viscous-damped servo system. However the coefficient $(L + F)$ must be substituted for the coefficient F of the earlier expressions. The relation is then obtained;

$$\theta = \epsilon^{-at}(B_1 \cos bt + B_2 \sin bt) \tag{6.11}$$

in which

$$\begin{cases} a = \dfrac{(L + F)}{2J} & (6.12) \\ b = \sqrt{\dfrac{K}{J} - \dfrac{(L + F)^2}{4J^2}}. & (6.13) \end{cases}$$

Complete Solution of the Equation.—The final step in obtaining a complete solution of Eq. (6.3) when the input angle θ_i is a step function of time, as given in Eqs. (6.4) and (6.5), is to combine the steady-state solution, Eq. (6.9), and the transient solution, Eq. (6.11), and express the conditions of the servo system at the instant when the transient is initiated. Thus, adding the steady-state and transient solutions,

$$\theta = \frac{F\omega_1}{K} + \epsilon^{-at}(B_1 \cos bt + B_2 \sin bt). \tag{6.14}$$

This Eq. (6.14) is the same as the previously derived equation (4.33), except for the values of the parameters a and b. Moreover, the boundary conditions of the system being the same as in the previous cases, the constants B_1 and B_2 are determined in exactly the same manner as described in Chap. IV, Eqs. (4.34) to (4.45) inclusive. This identity of boundary conditions and of the formal Eqs. (4.33) and (6.14) makes it possible to write at once

$$\begin{cases} B_1 = -\dfrac{F\omega_1}{K} & (6.15) \\ B_2 = \dfrac{1}{b}\left(\omega_1 - \dfrac{Fa\omega_1}{K}\right). & (6.16) \end{cases}$$

These expressions differ from Eqs. (4.40) and (4.45) in the actual values of a and b, given here in Eqs. (6.12) and (6.13). Equation (6.14) then becomes

$$\theta = \frac{F\omega_1}{K} + \epsilon^{-at}\left[-\frac{F\omega_1}{K}\cos bt + \frac{1}{b}\left(\omega_1 - \frac{Fa\omega_1}{K}\right)\sin bt\right]. \tag{6.17}$$

Undamped System.—In the limiting case where both damping factors L and F are zero, Eqs. (6.12) and (6.13) become

$$\begin{cases} a = 0 & (6.18) \\ b = \sqrt{\dfrac{K}{J}}. & (6.19) \end{cases}$$

Substituting these values in Eq. (6.17), there comes

$$\theta = \frac{\omega_1}{\sqrt{\frac{K}{J}}} \sin \sqrt{\frac{K}{J}}\, t. \tag{6.20}$$

This equation is exactly the same as Eqs. (4.49) and (5.21), which express the error function of, respectively, a viscous-damped servo with zero damping constant F and an error-rate-damped servo with zero damping constant L. As in these two previous cases, the error initiated by the step input function is an undamped sinusoidal function of time, which continues indefinitely. The radian frequency of this oscillation, or natural frequency of the servo, is

$$\omega_n = \sqrt{\frac{K}{J}}, \tag{6.21}$$

as previously defined in Eq. (4.50).

Critically Damped System.—Following the same procedure as in Chap. IV, Eq. (6.17) is rewritten

$$\theta = \frac{F\omega_1}{K} - \epsilon^{-at}\left[\frac{F\omega_1}{K}\cos bt + \left(\frac{Fa\omega_1}{K} - \omega_1\right)\frac{\sin bt}{b}\right]. \tag{6.22}$$

As b is made to approach zero, the terms $\cos bt$ and $(\sin bt)/b$ tend, respectively, toward 1 and t

$$b \to 0 \begin{cases} \cos bt \to 1 & (6.23) \\ \dfrac{\sin bt}{b} \to t. & (6.24) \end{cases}$$

Substituting these values in Eq. (6.22), the error is expressed

$$\theta = \frac{F\omega_1}{K} - \epsilon^{-at}\left[\frac{F\omega_1}{K} + \left(\frac{Fa\omega_1}{K} - \omega_1\right)t\right] \tag{6.25}$$

which contains no periodic (sine or cosine) term. Thus, in this limiting case, the error approaches its steady-state value without oscillation.

As stated above, this condition known as *critical damping* obtains when b tends toward zero; or, from Eq. (6.13), when

$$\frac{(L + F)^2}{4J^2} \to \frac{K}{J}. \tag{6.26}$$

It follows that the damping required for critical damping is equal to

$$(L + F)_c = 2\sqrt{KJ}. \tag{6.27}$$

Underdamped System.—For systems with different damping than that required for the critically damped condition, a parameter c may be used,

similar to that previously defined in Eq. (4.57). This parameter is the ratio of the actual damping of the system to that necessary for critical damping.

$$c = \frac{(L + F)}{(L + F)_c}. \tag{6.28}$$

Combining Eqs. (6.27) and (6.28),

$$(L + F) = 2c\sqrt{KJ}. \tag{6.29}$$

Inspection of Eq. (6.11) and its auxiliary equations (6.12) and (6.13) shows that the transient error function is dependent on the sum $(L + F)$ of the viscous damping and error-rate damping. On the other hand, Eq. (6.9) shows that the steady-state error is only a function of the viscous damping F. It is convenient to introduce a new parameter r to denote the ratio of the viscous damping alone to the combination of viscous damping and error-rate damping.

$$r = \frac{F}{L + F}. \tag{6.30}$$

Substituting Eqs. (6.21) and (6.29) in Eqs. (6.12) and (6.13) gives, as in the case of a viscous-damped system,

$$\begin{cases} a = c\omega_n & (6.31) \\ b = \omega_n \sqrt{1 - c^2} & (6.32) \end{cases}$$

Combining Eqs. (6.21), (6.29), and (6.30),

$$\frac{F}{K} = \frac{2rc}{\omega_n}. \tag{6.33}$$

Dimensionless Form of Equations.—Inserting Eqs. (6.31) and (6.33) in the *critically damped* equation (6.25), for which the damping ratio c is equal to unity,

$$\theta = \frac{2r\omega_1}{\omega_n} - \epsilon^{-\omega_n t}\left[\frac{2r\omega_1}{\omega_n} + (2r\omega_1 - \omega_1)t\right] \tag{6.34}$$

or

$$\theta\frac{\omega_n}{\omega_1} = 2r - [2r + (2r - 1)\omega_n t]\epsilon^{-\omega_n t}. \tag{6.35}$$

Similarly, substituting Eqs. (6.31), (6.32), and (6.33) in the *general oscillatory* equation (6.17), this becomes

$$\theta = \frac{2rc\omega_1}{\omega_n} - \epsilon^{-c\omega_n t}\left[\frac{2rc\omega_1}{\omega_n}\cos\omega_n\sqrt{1 - c^2}\,t + \left(\frac{2rc^2\omega_1}{\omega_n\sqrt{1 - c^2}} - \frac{\omega_1}{\omega_n\sqrt{1 - c^2}}\right)\sin\omega_n\sqrt{1 - c^2}\,t\right] \tag{6.36}$$

or

$$\theta \frac{\omega_n}{\omega_1} = 2rc - \left[2rc \cos \omega_n \sqrt{1 - c^2}\, t + \frac{2rc^2 - 1}{\sqrt{1 - c^2}} \sin \omega_n \sqrt{1 - c^2}\, t\right] \epsilon^{-c\omega_n t}. \tag{6.37}$$

Equations (6.35) and (6.37) express in dimensionless form the error of a combined viscous-damped and error-rate-damped servomechanism in which the input member is suddenly set in motion at constant speed.

Equation (6.37) applies for values of the damping ratio c comprised between zero and unity. For $c > 1$, as occurs when the system is overdamped, this equation may be written in the form

$$\theta \frac{\omega_n}{\omega_1} = 2rc - \left[2rc \cosh \omega_n \sqrt{c^2 - 1}\, t + \frac{2rc^2 - 1}{\sqrt{c^2 - 1}} \sinh \omega_n \sqrt{c^2 - 1}\, t\right] \epsilon^{-c\omega_n t}. \tag{6.37a}$$

When applying these equations, the units used for expressing θ, ω_1, ω_n, and t must be so chosen as to make the factors $\theta(\omega_n/\omega_1)$ and $\omega_n t$ dimensionless, as explained before in Chap. IV.

Mechanical Analogy.—Before the equations obtained in the preceding paragraphs are discussed, it may be helpful to describe a purely mechanical system embodying both viscous and error-rate damping, like the servo systems considered here.

In the preceding chapter, Fig. 5.3 and 5.4 illustrate, respectively, a viscous-damped and an error-rate-damped pendulum. The system of Fig. 6.2 combines the features of these two pendulums. It comprises a mass M suspended from a pivot O by a thin rigid rod. The pivot is mounted on a carriage C that can be displaced along a horizontal rail RR'. The mass M is provided with a vane V, which dips in a damping fluid, such as water or oil, contained in a stationary tank T. A second vane V' is mounted on the pendulum rod and dips in a damping fluid contained in a tank T' supported by the carriage C.

Although this system is not a servomechanism, it may be likened to one, under the conditions described below. The carriage and pivot may then be considered as equivalent to the input member of a servo, while the pendulum mass is taken as representing the output member.

Suppose that the carriage is moving at constant speed from left to right, as shown by the arrow A, and that this motion has been sustained for a considerable time, so that all transients of the starting period will have died out. Under such steady-state conditions, the pendulum assumes a slanting position, such as shown in dotted line in Fig. 6.2, because of the retarding viscous friction force developed between the vane V and the damping liquid in the stationary tank T. This was

explained in relation to Fig. 5.3. The resulting position error E between the mass M and pivot O is proportional to the speed of translatory motion of the system and to the amount of output friction.

On the other hand, at the time the carriage is suddenly started from rest, or when after it has been set in motion the system is suddenly stopped, the pendulum will oscillate. The friction forces then arising

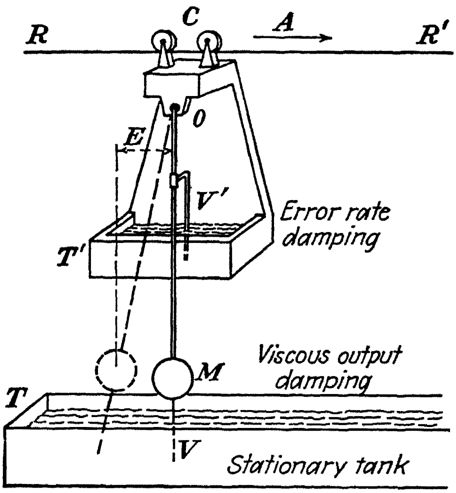

FIG. 6.2.—Pendulum analogy of servomechanism with combined viscous output damping and error-rate damping.

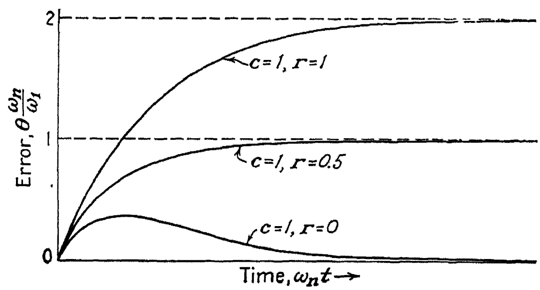

FIG. 6.3.—Dimensionless error-time curves for critically damped servomechanism with combined viscous output damping and error-rate damping.

between the vane V' and the liquid in the tank T' and between the vane V and the liquid in the tank T will both damp out the oscillatory motion.

It follows that if in the system of Fig. 6.2 the viscous damping afforded by the vane V and the liquid in the tank T is small enough to produce a tolerably small steady-state error E but too small to damp out quickly the transient oscillation or to keep its amplitude (overshoot) down to a tolerable value, the oscillation damping may be increased as desired by

increasing the error-rate damping obtained from the vane V' in the tank T'. Such additional error-rate damping will not increase the steady-state error, and it can be adjusted independently from the viscous damping produced by the vane V in the tank T.

Discussion of Dimensionless Equations.—It is apparent from the preceding discussion that a variety of operating conditions may be encountered, depending on (1) the total damping $(F + L)$ and (2) the relative amounts of viscous damping F and error-rate damping L. In other words, the performance is dependent on the values of the damping ratios c and r defined above in Eqs. (6.28) and (6.30), respectively. This may be illustrated by representing graphically Eqs. (6.35), (6.37), and (6.37a) which express the error $\theta(\omega_n/\omega_1)$ as a function of time $\omega_n t$. The Eq. (6.35) corresponds to a critically damped system in which $c = 1$, while

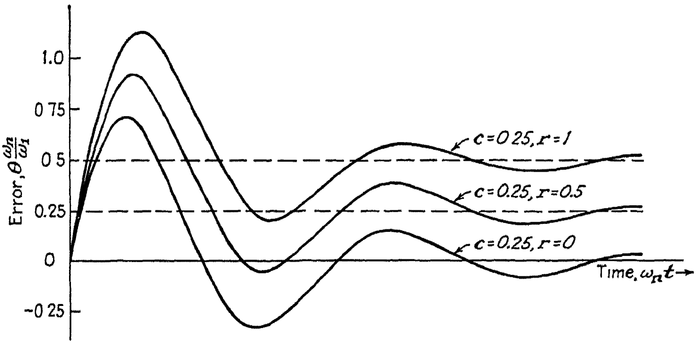

FIG. 6.4.—Dimensionless error-time curves for underdamped servomechanism with combined viscous output damping and error-rate damping.

the relation, Eq. (6.37), corresponds to an underdamped system for which $c < 1$, and Eq. (6.37a) corresponds to the overdamped case in which $c > 1$. In any of these cases, the error-time curve may be calculated and plotted for any desired value of the ratio r, as shown in Figs. 6.3 and 6.4. It will be noted that for $r = 1$ there is no error-rate damping, and the system is purely viscous damped. For $r = 0$ there is no viscous damping, and the system has only error-rate damping.

The curves show also that for any given value of c the steady-state error decreases with the value of r, while changing the value of r only does not alter the duration of the transient.

Problem.—A friction-damped servomechanism with known controller constant K, output friction F, and natural resonant frequency ω_n is found to have four times the allowable steady-state error. The servo is brought within limits, without changing the relative damping ratio c, by the addition of error-rate damping and a change in the controller gain. Determine the change in controller constant, the change in the natural resonant frequency, and the relative amount of error-rate damping constant L to the original output damping F already in the system.

Solution: In any servomechanism with viscous output damping, the output torque $K\theta$ of the controller must, in the steady-state condition, equal the output-friction force $F\omega_1$, where ω_1 is the output speed of the system.

$$K\theta = F\omega_1$$

from which

$$K = \frac{F\omega_1}{\theta}.$$

This relation was given before in Eq. (6.9)

It follows that if the error is to be reduced to one-fourth its original value, the controller factor must be made four times larger, and therefore becomes

$$K' = 4K.$$

But from the relation, Eq. (6.21), the natural frequency ω_n of the system then changes from

$$\omega_n = \sqrt{\frac{K}{J}}$$

to

$$\omega_n = \sqrt{\frac{K'}{J}} = \sqrt{\frac{4K}{J}} = 2\sqrt{\frac{K}{J}} = 2\omega_n.$$

Thus, by increasing the controller factor K to a value $K' = 4K$, as required to reduce the error to one-quarter its original value, the natural frequency of the system is automatically doubled.

In order to find how much error-rate damping must be added to obtain the desired result, reference is made to the relation, Eq. (6.37), which shows the steady-state error to be

$$\theta\frac{\omega_n}{\omega_1} = 2rc,$$

from which

$$r = \theta\frac{\omega_n}{\omega_1}\frac{1}{2c},$$

where, by definition Eq. (6.30),

$$r = \frac{F}{L + F}.$$

In the original system

$$L = 0,$$

and

$$r = \theta\frac{\omega_n}{\omega_1}\frac{1}{2c} = 1.$$

In the new system with added error-rate damping, it was found that the error is only $\theta/4$, while the natural frequency is $2\omega_n$. Thus

$$r' = \frac{\theta}{4}\frac{2\omega_n}{\omega_1}\frac{1}{2c} = \frac{1}{2}r = \frac{1}{2};$$

or

$$\frac{F}{L + F} = \frac{1}{2},$$

from which

$$L = F.$$

Response to Sinusoidal Input Function.—The steady-state frequency response of a servo system with combined viscous output damping and error-rate damping can be found by the same method as was applied in the simpler cases previously studied in Chap. IV and V. First, the differential equation (6.1) of the system is expressed in terms of the input and output angular positions. Then, a sinusoidal input function is written in this equation. Since the system is linear, the output function is of similar form, and differs from the input function only in amplitude and phase. These are determined in terms of the input frequency and convenient system parameters, and are expressed in dimensionless form suitable for practical applications.

The equation of motion of a servomechanism with combined viscous output damping and error-rate damping was established at the beginning of this chapter as

$$K\theta + L\frac{d\theta}{dt} = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt} \tag{6.1}$$

where

$$\theta = \theta_i - \theta_o. \tag{6.2}$$

Eliminating the error θ between these two equations, a relation is obtained between the input and output displacements,

$$K\theta_i + L\frac{d\theta_i}{dt} = J\frac{d^2\theta_o}{dt^2} + (L + F)\frac{d\theta_o}{dt} + K\theta_o. \tag{6.38}$$

If in this linear equation the input angular position θ_i is a unit complex sinusoidal function of time

$$\theta_i = \epsilon^{j\omega t} \tag{6.39}$$

then the output position θ_o is of the form

$$\theta_o = A\epsilon^{j(\omega t+\lambda)}. \tag{6.40}$$

The same values of A and λ may be obtained by using a real sinusoidal input function of unit amplitude.

$$\theta_i = \cos \omega t \tag{6.41}$$

the output function being then

$$\theta_o = A \cos (\omega t + \lambda). \tag{6.42}$$

The validity of this transformation was discussed in Chap. IV.

The first time derivative of the input function, Eq. (6.39), is

$$\frac{d\theta_i}{dt} = j\omega\epsilon^{j\omega t} \tag{6.43}$$

while the first and second time derivatives of the output function, Eq. (6.40) are

$$\frac{d\theta_o}{dt} = j\omega A\epsilon^{j(\omega t+\lambda)} \tag{6.44}$$

$$\frac{d^2\theta_o}{dt^2} = -\omega^2 A\epsilon^{j(\omega t+\lambda)}. \tag{6.45}$$

Substitution of Eqs. (6.39), (6.40), (6.43), (6.44), and (6.45) in Eq. (6.38) gives

$$K\epsilon^{j\omega t} + j\omega L\epsilon^{j\omega t} = -\omega^2 JA\epsilon^{j(\omega t+\lambda)} + j\omega(L+F)A\epsilon^{j(\omega t+\lambda)} + KA\epsilon^{j(\omega t+\lambda)} \tag{6.46}$$

Dividing through by $\epsilon^{j\omega t}$,

$$K + j\omega L = -\omega^2 JA\epsilon^{j\lambda} + j\omega(L+F)A\epsilon^{j\lambda} + KA\epsilon^{j\lambda} \tag{6.47}$$

or

$$A\epsilon^{j\lambda} = \frac{K + j\omega L}{K - \omega^2 J + j\omega(L+F)}. \tag{6.48}$$

This last equation is an expression of the output function vector, as referred to the input function vector. Dividing the numerator and denominator of the right-hand member by K, this expression becomes

$$A\epsilon^{j\lambda} = \frac{1 + j\omega\dfrac{L}{K}}{1 - \omega^2\dfrac{J}{K} + j\omega\dfrac{L+F}{K}}. \tag{6.49}$$

Substituting the values of L/K, J/K, and $(L+F)/K$, as obtained from Eq. (6.21), (6.29), and (6.30),

$$\frac{L}{K} = \frac{2c(1-r)}{\omega_n}, \qquad \frac{J}{K} = \frac{1}{\omega_n^2}, \qquad \frac{L+F}{K} = \frac{2c}{\omega_n},$$

Eq. (6.49) is written

$$A\epsilon^{j\lambda} = \frac{1 + j2c(1-r)\dfrac{\omega}{\omega_n}}{1 - \dfrac{\omega^2}{\omega_n^2} + j2c\dfrac{\omega}{\omega_n}}. \tag{6.50}$$

This can be expressed more simply by introducing the variable d to denote the relative operating frequency.

$$d = \frac{\omega}{\omega_n}. \tag{6.51}$$

Equation (6.50) then becomes

$$A\epsilon^{j\lambda} = \frac{1 + j2c(1-r)d}{1 - d^2 + j2cd}. \tag{6.52}$$

Equation (6.50), or its equivalent form Eq. (6.52), may be considered as representing a vector of magnitude A and phase angle λ (both relative to the input displacement taken as a unit reference vector) equal, respectively, to

$$\begin{cases} A = \sqrt{\dfrac{1 + 4c^2(1-r)^2 \dfrac{\omega^2}{\omega_n^2}}{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + 4c^2 \dfrac{\omega^2}{\omega_n^2}}} = \sqrt{\dfrac{1 + 4c^2d^2(1-r)^2}{(1-d^2)^2 + 4c^2d^2}} & (6.53) \\ \lambda = \tan^{-1} 2c(1-r)\dfrac{\omega}{\omega_n} - \tan^{-1} \dfrac{2c\dfrac{\omega}{\omega_n}}{1 - \dfrac{\omega^2}{\omega_n^2}} \\ \quad = \tan^{-1} 2cd(1-r) - \tan^{-1} \dfrac{2cd}{1-d^2}. & (6.54) \end{cases}$$

For an input function of the form given in Eq. (6.41)

$$\theta_i = \cos \omega t, \tag{6.41}$$

the output function of the servo system is found by substituting Eqs. (6.53) and (6.54) in Eq. (6.42),

$$\theta_o = \sqrt{\frac{1 + 4c^2d^2(1-r)^2}{(1-d^2)^2 + 4c^2d^2}} \cos\left(\omega t + \tan^{-1} 2cd(1-r) - \tan^{-1}\frac{2cd}{1-d^2}\right) \tag{6.55}$$

This last equation expresses, as a function of time, the instantaneous position θ_o of the output member of a servo system with combined viscous output damping and error-rate damping, when the input member is displaced back and forth according to the sinusoidal time function, Eq. (6.41).

From the Eqs. (6.53) and (6.54), the relative amplitude A and phase angle λ of the output function may be calculated and plotted as functions of the relative input frequency d for given values of the damping ratios c and r. The resulting resonance curves differ somewhat from those found previously for servo systems having *either* viscous output damping *or* error-rate damping. In particular, the curves showing the amplitude A as a function of the relative frequency d will cross, unlike those of Fig. 4.11, which meet only at the point of origin. However, unlike the curves of Fig. 5.10, the curves cross at various points, instead of crossing all at one point. The curves are not plotted here, in view of the great variety of possible combinations of the values of c and r.

CHAPTER VII

ERROR-RATE STABILIZATION NETWORKS

The last two chapters were concerned with an analysis of the properties of error-rate-damped (or stabilized) servo systems. However, only passing reference was made, in Chap. V, to the practical means employed for obtaining this type of damping. This question will now be taken up in greater detail.

As set forth previously, the error voltage signal that actuates the controller (amplifier and servo motor) may be derived either from a source of direct voltage associated with a potentiometer, or from a source of alternating voltage associated with synchro repeater devices.

Stabilization of a servo system was shown to be obtainable through viscous output damping or through error-rate damping. Viscous output damping is achieved by including in the system a friction damper, or an eddy-current damper, or by using a servo motor that has a suitable characteristic of diminishing torque with increasing speed. On the other hand, error-rate damping may be obtained by inserting in the system an error signal differentiating device. An example of such a device was described in relation to Fig. 5.5, as applied to a system where the error signal is obtained in the form of a direct voltage.

However, irrespective of whether the input-output position error is converted into a direct or an alternating voltage to be fed into the controller, and irrespective of what particular error voltage differentiating device is used (error-rate detecting device), an error-rate-damped servo system is characterized by the fact that the controller output torque is a function of both the error and the rate of change of the error. The form of this function will be determined in the following paragraph, before studying the electrical networks through which it may actually be applied in practice.

Dependence of Torque on Error Rate of Change.—In the preceding chapters it was shown, Eqs. (5.1) and (6.1), that the torque produced by the controller in an error-rate-damped servo system is equal to

$$T = K\theta + L\frac{d\theta}{dt} \tag{7.1}$$

where $K\theta$ and $L(d\theta/dt)$ are, respectively, the error and error-rate components of the torque. Thus, instead of being proportional to the error alone, as in a viscous-damped servo system, the torque is here also a function of the time rate of change of the error.

In practice, the error and its rate of change may be any functions of time. However, complicated as these functions may be, they can always be expressed as a sum of sinusoidal functions of time through application of the methods of Fourier analysis. In order to form an idea of the manner in which the controller torque varies with the rate of change of the error, it is therefore permissible to consider the simple particular case where the input member of the system is moved in such fashion that the error is a sinusoidal function of time of unit amplitude.

$$\theta = \sin \omega t. \tag{7.2}$$

The first time derivative of the error being then

$$\frac{d\theta}{dt} = \omega \cos \omega t, \tag{7.3}$$

the above Eq. (7.1) of the torque per unit error then becomes[1]

$$\begin{aligned} T &= K \sin \omega t + \omega L \cos \omega t \\ &= \sqrt{K^2 + \omega^2 L^2} \sin (\omega t + \lambda) \\ &= K \sqrt{1 + \frac{\omega^2 L^2}{K^2}} \sin (\omega t + \lambda). \end{aligned} \tag{7.4}$$

Thus, when the error θ is a sinusoidal function of time, the torque T is also a sinusoidal function of time and leads the error by a phase angle λ equal to

$$\lambda = \tan^{-1} \frac{\omega L}{K}. \tag{7.5}$$

The Eqs. (7.4) and (7.5) for the torque and its phase angle may be written more simply by substituting in these the symbol ω_b for the ratio K/L.

$$\frac{K}{L} \equiv \omega_b. \tag{7.6}$$

The above expressions Eqs. (7.4) and (7.5), then become

$$\left\{ \begin{aligned} T &= K\sqrt{1 + \frac{\omega^2}{\omega_b{}^2}} \sin(\omega t + \lambda) &\quad (7.7) \\ \lambda &= \tan^{-1} \frac{\omega}{\omega_b}. &\quad (7.8) \end{aligned} \right.$$

[1] The expression is of the form

$$A \sin x + B \cos x,$$

which, according to a theorem of trigonometry, can be written

$$\left\{ \begin{aligned} &\sqrt{A^2 + B^2} \sin (x + \lambda) \\ &\lambda = \tan^{-1} \frac{B}{A}. \end{aligned} \right.$$

The amplitude of the torque per unit error is thus

$$T_m = K\sqrt{1 + \frac{\omega^2}{\omega_b^2}} \tag{7.9}$$

and varies with the error frequency ω. For very low frequencies, the torque amplitude is equal to K, since ω is then approximately zero. As the error variation rate or frequency increases, the torque amplitude increases also, and tends asymptotically toward the expression $K\omega/\omega_b$, as shown in the upper graph of Fig. 7.1. It will be noted that for ω

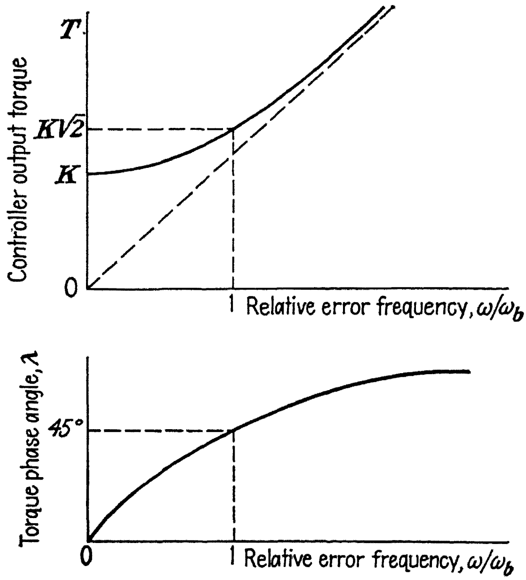

FIG. 7.1.—Torque amplitude and phase angle as functions of frequency in error-rate-damped servomechanism.

$= \omega_b$ the torque leads the error by 45 deg. and has a magnitude $K\sqrt{2}$, or $\sqrt{2}$ times the zero frequency torque.[1]

Any servo system that has a torque-frequency response characteristic similar to that described above (torque increasing with error frequency, in the manner shown) is equivalent to an error-rate-damped system and has similar operating properties. Practical means for obtaining this type of response in a servo system will be discussed further below.

[1] If the error function, Eq. (7.2), is written in the complex exponential form

$$\theta = \epsilon^{j\omega t} \tag{7.2a}$$

the expression for the output torque can be written

$$T = K\left(1 + j\frac{\omega}{\omega_b}\right)\epsilon^{j\omega t}. \tag{7.7a}$$

Equation (7.6) may be rewritten in terms of the system parameters defined in the preceding chapters, *viz.*,

$$\text{Natural frequency of servo system} = \omega_n = \sqrt{\frac{K}{J}} \tag{7.10}$$

$$\text{Ratio of viscous output damping to total damping} = r = \frac{F}{L + F} \tag{7.11}$$

$$\text{Ratio of actual damping to critical damping} = c = \frac{L + F}{(L + F)_c}. \tag{7.12}$$

For this purpose, Eq. (6.27) of the preceding chapter and Eq. (7.12) are combined and solved for L.

$$L = 2c\sqrt{KJ} - F. \tag{7.13}$$

Substituting this relation in Eq. (7.6) gives

$$\omega_b = \frac{K}{2c\sqrt{KJ} - F}. \tag{7.14}$$

Writing F as specified in Eq. (7.11) in Eq. (7.14), this becomes

$$\omega_b = \frac{K}{2c\sqrt{KJ} - r(L + F)} \tag{7.15}$$

and substituting Eq. (7.13) in Eq. (7.15) gives

$$\omega_b = \frac{K}{2c\sqrt{KJ}\,(1 - r)} = \sqrt{\frac{K}{J}}\,\frac{1}{2c(1 - r)}. \tag{7.16}$$

Finally, by writing Eq. (7.10) in Eq. (7.16),

$$\omega_b = \frac{\omega_n}{2c(1 - r)}, \tag{7.17}$$

an equation is obtained which expresses the error frequency at which the torque response is $\sqrt{2}$ times that corresponding to a constant error (zero frequency).

A form of r convenient for use with Eq. (7.17) is obtained by combining Eqs. (7.10), (7.11), (7.12) and (6.27).

$$r = \frac{F}{2c\omega_n J}. \tag{7.18}$$

Problem.—A servo system has inertia J and viscous damping F which, referred to the motor shaft, have values of

$$J = 16 \times 10^{-6} \text{ slug-ft.}^2$$
$$F = 72 \times 10^{-6} \text{ ft.-lb. per radian per sec.}$$

Calculate the elements of the servo and the steady-state error at 10 r.p.m. for a damping ratio $c = 0.4$, first assuming the system to have viscous damping only; then assuming the natural frequency to be increased to four times its original value and error-rate damping to be added so as to hold the damping ratio c to its original value of 0.4.

Solution: If the system has only viscous output damping, its natural frequency is [from Eqs. (4.50) and (4.58)]

$$\omega_n = \frac{F}{2cJ} = \frac{72 \times 10^{-6}}{2 \times 0.4 \times 16 \times 10^{-6}} = 5.6 \text{ radians per sec.}$$
$$= 0.9 \text{ cycle per sec.}$$

At a speed of 10 r.p.m. (or approximately 1 radian per sec.), the steady-state error will be

$$\theta \frac{\omega_n}{\omega_1} = 2c \tag{4.65}$$

or

$$\theta = \frac{2c\omega_1}{\omega_n} = \frac{2 \times 0.4 \times 1}{5.6} \times 57.3 = 8 \text{ deg. (approximately).}$$

As shown by the expression $\theta = (2c\omega_1/\omega_n)$, this large error can be reduced by either decreasing the value of c or increasing the value of ω_n. Decreasing the value of c would increase the amplitude and duration of the transient oscillation, and will therefore not be considered here. Without changing c, the natural frequency ω_n will then be raised by introducing error-rate damping into the system. Let ω_n be increased to four times its original value of 5.6 radians per sec. Using Eq. (7.18) for calculating the damping ratio r defined by the relation, Eq. (6.30),

$$r = \frac{F}{2c\omega_n J} = \frac{72 \times 10^{-6}}{2 \times 0.4 \times 5.6 \times 4 \times 16 \times 10^{-6}} = 0.25,$$

it becomes possible to determine the error frequency ω_b [from Eq. (7.17)] for which the torque of the error-rate-damped servo is $\sqrt{2}$ times greater than for zero error frequency.

$$\omega_b = \frac{\omega_n}{2c(1 - r)} = \frac{5.6 \times 4}{2 \times 0.4 \times (1 - 0.25)} = 37.5 \text{ radians per sec.}$$

Applying Eq. (6.37), the steady-state error at a speed ω_1 of 10 r.p.m. (or approximately 1 radian per sec.) is equal to

$$\theta \frac{\omega_n}{\omega_1} = 2rc$$

or

$$\theta = 2rc \frac{\omega_1}{\omega_n} = 2 \times 0.25 \times 0.4 \times \frac{1}{5.6 \times 4} \times 57.3 = 0.5 \text{ deg.,}$$

instead of 8 deg. found before the addition of error-rate damping.

It should be noted that the inclusion of error-rate damping has raised the natural frequency ω_n from 5.6 to 5.6 × 4, or 22.4 radians per sec. In view of the relation

$$\omega_n = \sqrt{\frac{K}{J}} \tag{7.10), (4.50}$$

this fourfold increase of ω_n requires a sixteenfold increase of the controller gain K.

Correspondingly, in view of the relation, Eqs. (6.9) and (4.46),

$$K\theta = F\omega_1 \qquad \text{or} \qquad \theta = \frac{F\omega_1}{K}$$

and of the fact that the coefficient F remained constant, this sixteenfold increase results in a sixteenfold reduction of the error θ. This agrees with the calculated values, whereby the error was found to be reduced from 8 deg. to 0.5 deg.

If the steady-state error θ is specified in the problem, the natural frequency ω_n is readily calculated from the relations previously given

$$\theta \frac{\omega_n}{\omega_1} = 2rc \qquad \text{or} \qquad \omega_n = 2rc \frac{\omega_1}{\theta} \tag{6.37}$$

$$r = \frac{F}{2c\omega_n J} \qquad \text{(from Eqs. 6.21 and 6.33)}$$

from which

$$\omega_n = \sqrt{\frac{\omega_1}{\theta} \frac{F}{J}} \tag{7.19}$$

where the ratio ω_1/θ is the velocity figure of merit of the system, as defined in Chap. IV.

Direct-current Networks for Error-rate Stabilization.—When the device for converting the input-output error angle into an electrical voltage is energized by a source of continuous potential, as was shown in Fig. 2.3 of Chap. II, for example, an error-rate detecting device may be added to the system for supplying in addition to the continuous error voltage a voltage that is proportional to the speed, or variation rate, of the error.

Such an arrangement was described in Chap. V and illustrated in Fig. 5.5, where a generator driven from the differential error shaft provides a voltage that is proportional to the rate of change of the error. This error-rate voltage was injected in series with the error voltage developed by the differential potentiometer, and the resulting combined error and error-rate voltage was applied to the input terminals of the controller amplifier.

Instead of a generator, a so-called *differentiating network* may be inserted in the error voltage channel to produce the error-rate voltage. Such a network is often preferred to the generator, since it has no moving parts and is relatively simple to build. The system is illustrated in Fig. 7.2. It differs from the arrangement of Fig. 5.5 in that the potentiometer slider P and center tap of the battery B, instead of being connected to the amplifier terminals through the generator G, are connected to these terminals by way of a resistance-capacity network CR_1R_2.

If the input-output error is constant, the potentiometer slider P assumes some fixed position corresponding to the particular error value. The resulting constant potentiometer voltage e_i causes a constant current to flow in the resistor circuit branch R_1R_2, and produces across R_1 a

voltage proportional to the error voltage e_i. This proportional voltage applied to the input terminals of a d-c amplifier energizes the servo motor after it has been amplified. The capacitor C, connected across the resistor R_2, is charged to the potential difference produced between the terminals of this resistor by the current flowing in the circuit R_1R_2 and plays no part in the process as long as the error remains constant.

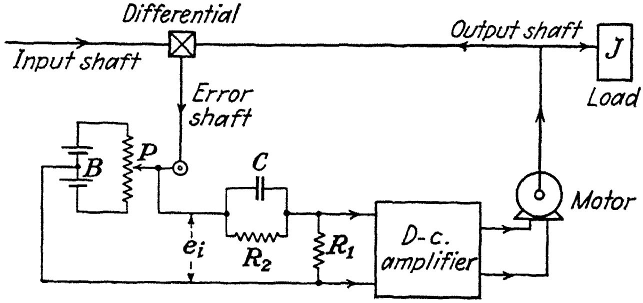

FIG. 7.2.—Error-rate damped servomechanism with d-c error signal and differentiating network.

However, if the error changes to some other value, corresponding changes occur in the positions of the differential error shaft and of the potentiometer slider P, as well as in the potentiometer voltage e_1 and the current in the circuit branch R_1R_2. The current variation, in

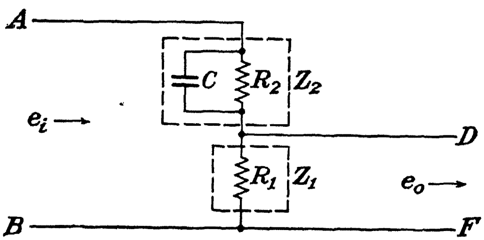

FIG. 7.3.—Differentiating network.

turn, produces proportional variations of the voltage drops across the resistors R_1 and R_2, respectively. As the voltage across R_2 varies, the electric charge on the capacitor C must vary accordingly, in order that the capacitor voltage may at all times remain equal to the voltage drop across R_2. The resulting capacitor charging (or discharging) current is proportional to the *rate of change* of the capacitor voltage; hence it is proportional also to the rate of change of the differential error. This current in flowing through the circuit adds to or subtracts from the current produced by the potentiometer error voltage e_i. Thus, the voltage drop across R_1 is proportional to the algebraic sum of the error and time rate of change of the error.

To analyze quantitatively the operation of the differentiating network just described, this is redrawn separately in Fig. 7.3. The input voltage, which will be designated as e_i, applied to the input terminals A and B of the network, is the instantaneous error voltage supplied by the battery and potentiometer device employed for converting error angle into electrical voltage. Its output voltage e_o, which appears at terminals D and F of the network, is fed to the controller amplifier and thence to the servo motor.

Considering the network as a voltage divider, the ratio of output voltage to input voltage is equal to the ratio of the impedance Z_1 of the output resistor R_1 to the total impedance $(Z_1 + Z_2)$ of the entire network.

$$\frac{e_o}{e_i} = \frac{Z_1}{Z_1 + Z_2} \tag{7.20}$$

or

$$\frac{e_o}{e_i} = \frac{R_1}{\dfrac{1}{\dfrac{1}{R_2} + j\omega C} + R_1} = \frac{1 + j\omega R_2 C}{\dfrac{R_2}{R_1} + 1 + j\omega R_2 C} \tag{7.21}$$

Introducing two parameters M and ω_b which will be helpful here and later on in the study of this and other error-rate networks[1]

$$M = \frac{R_2}{R_1} \tag{7.22}$$

and

$$\omega_b = \frac{1}{R_2 C} \tag{7.23}$$

Eq. (7.21) can be rewritten

$$\frac{e_o}{e_i} = \frac{1 + j(\omega/\omega_b)}{M + 1 + j(\omega/\omega_b)} \tag{7.24}$$

Under the condition that M is large (say, greater than 3) and that ω/ω_b is small with respect to $(M + 1)$, Eq. (7.24) may be simplified.

$$\frac{e_o}{e_i} = \frac{1 + j(\omega/\omega_b)}{M + 1}. \tag{7.25}$$

That the network produces the correct output voltage to be used for error rate stabilization may be seen by comparing Eq. (7.25) to Eq. (7.7*a*) or to Eqs. (7.7) and (7.8), which specify the necessary network response. In other words, the real and imaginary parts of Eq. (7.25) are equal when the error frequency ω is equal to ω_b. The output voltage is then 45 deg. out of phase with the input voltage and has a magnitude

[1] As shown later, the parameter ω_b is the same as was defined previously in the relation, Eq. (7.6).

$\sqrt{2}$ times as large as that of the input voltage. This frequency ω_b was defined in Eq. (7.6) and may be calculated from Eq. (7.17). It follows that

$$\omega_b = \frac{K}{L} = \frac{1}{R_2C} = \frac{\omega_n}{2c(1-r)}. \quad (7.6),\ (7.17),\ \text{and}\ (7.23)$$

As long as Eq. (7.25) is valid for the network, the latter behaves as an error-rate damping network should. However, when the error frequency is large enough to make ω/ω_b comparable to $(M + 1)$, Eq. (7.24) must be used, and the network no longer produces the required error-plus-error-rate voltage.

Thus M should be chosen so that $(M + 1) > \omega/\omega_b$ for the maximum value of frequency ω that is expected to appear in the error signal. This maximum value of ω should be several times as large as ω_b for most practical applications. If M is made too large, the output-input voltage ratio e_o/e_i of the network becomes so small that an excessive amount of amplifier gain will be required to compensate for the voltage loss across the network. A rigorous solution of a system employing this network leads to a third-order differential equation, which will not be derived or discussed here, since the above considerations are generally sufficient to allow a satisfactory design calculation.

Problem.—The servo system of Fig. 7.2 comprises a continuous-voltage error translating device with battery B and potentiometer P, an error-rate detecting network CR_1R_2, a d-c amplifier and motor, an inertia load, and a differential device between the input and output members. There is no viscous damping in the system. Knowing that R_1 has one-ninth the resistance value of R_2 and that the capacitor C has a value of 0.1 mf., calculate R_1 and R_2 to give a pure error-rate damping ratio $c = 0.25$, the system having a natural frequency $\omega_n = 4$ cycles per sec.[1]

Solution: From the relation, Eq. (7.17), and noting that $r = 0$, since there is no viscous damping,

$$\omega_b = \frac{\omega_n}{2c} = \frac{4 \times 2\pi}{2 \times 0.25} = 50 \text{ radians per sec.}$$
$$= 8 \text{ cycles per sec.}$$

From Eq. (7.23)

$$\omega_b = 50 = \frac{1}{R_2C} = \frac{1}{R_2 \times 0.1 \times 10^{-6}};$$

or

$$R_2 = 200{,}000 \text{ ohms},$$

and

$$R_1 = \frac{R_2}{9} = 22{,}200 \text{ ohms}.$$

[1] From the curves of Fig. 5.7, it is seen that for $c = 0.25$ the transient error is almost fully damped out at a time $\omega_n t = 18$. Since $\omega_n = 4$ cycles per sec. $= 4 \times 2\pi$ radians per sec., this corresponds approximately to

$$t = \frac{18}{\omega_n} = \frac{18}{4 \times 2\pi} = 0.75 \text{ sec.},$$

which, in many cases, is a reasonably fast response.

Under these conditions,[1] a constant error voltage of 1 volt applied to the input terminals of the network CR_1R_2 produces a voltage of 0.1 volt at the input terminals of the controller amplifier. An alternating error voltage of 1 volt, with a frequency of 8 cycles per sec., produces a voltage of 0.14 volt at the amplifier input terminals.

Alternating-current Networks for Error-rate Stabilization.—When a-c devices, such as the synchro repeaters described in an earlier chapter, are used for translating the error angle into an electrical voltage, circuit networks having suitable frequency-response characteristics may be used for supplying to the controller a voltage representing the variation

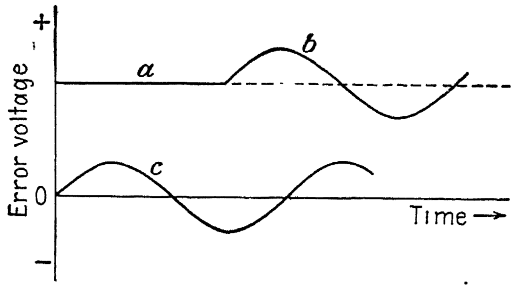

FIG. 7.4.—Error voltage produced by a d-c energized transducer.

rate of the error angle. Before a few of these networks are discussed, a clear understanding must be had of the nature of the error voltage developed by the synchro repeaters or similar devices.

In a d-c translating device, like the battery and potentiometer combination described before, a constant, unvarying error produces a constant, unvarying, proportional continuous voltage (see *a*, Fig. 7.4). A pulsating or alternating error of frequency ω produces, respectively, a proportional pulsating or alternating voltage of same frequency (see *b* and *c*, Fig. 7.4).

[1] According to Eq. (7.22), specifying that R_1 has one-ninth the value of R_2 can be written

$$M = \frac{R_2}{R_1} = 9$$

Now, it was stated that if ω_{max} is the highest error frequency encountered during operation, proper response of the network requires that

$$(M + 1) > \frac{\omega_{max}}{\omega_b} \qquad \text{or} \qquad (M + 1)\omega_b > \omega_{max}.$$

Since ω_b was found to be equal to 8 cycles per sec., this implies that

$$(9 + 1) \times 8 > \omega_{max}$$
$$80 \text{ cycles} > \omega_{max}$$

which, in the present application, is more than ample a margin.

With an a-c translating device, a constant error produces an alternating voltage

$$e = E_o \sin u_c t \tag{7.26}$$

of constant amplitude E_o proportional to the magnitude of the error and of frequency u_c that is the frequency of the a-c source energizing the device, 60 cycles, for example (see *a*, Fig. 7.5).

A pulsating error of frequency ω produces an alternating voltage

$$e = (E_o + E_1 \sin \omega t) \sin u_c t \tag{7.27}$$

having an amplitude $(E_o + E_1 \sin \omega t)$ proportional to the instantaneous value of the error and a frequency u_c, which is the frequency of the a-c source that energizes the device (see *b*, Fig. 7.5).

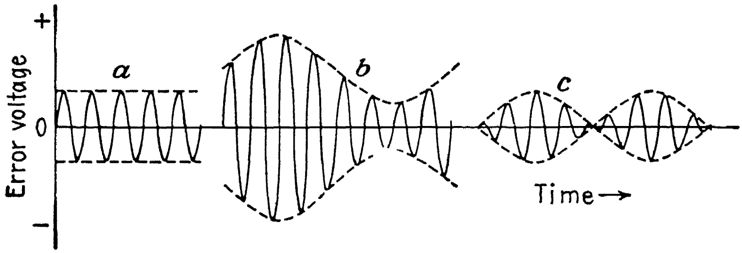

FIG. 7.5—Error voltage produced by an a-c energized transducer.

An alternating error of frequency ω produces an alternating voltage

$$e = (E_1 \sin \omega t) \sin u_c t, \tag{7.28}$$

which has an alternating amplitude $(E_1 \sin \omega t)$ proportional to the instantaneous value of the error and a frequency u_c which, again, is the frequency of the source which energizes the device (see *c*, Fig. 7.5).

In other words, a pulsating or alternating error modulates at the error frequency ω the amplitude of the alternating voltage of *line* or *carrier* frequency u_c energizing the device. A simple trigonometric transformation allows Eq. (7.28) to be written in the form

$$e = \tfrac{1}{2}E_1[\cos(u_c - \omega)t - \cos(u_c + \omega)t]. \tag{7.29}$$

This shows that with a translating device energized by an alternating current source of frequency u_c an alternating error of frequency ω produces two alternating error voltages. The frequencies of these two voltages are equal, respectively, to the sum and the difference of the frequencies of the error and of the energizing alternating voltage.

These two error voltage frequencies $(u_c - \omega)$ and $(u_c + \omega)$ thus differ from the fixed line or carrier frequency u_c by the error, or modulation, frequency ω. An error voltage of carrier frequency u_c therefore corre-

sponds to a constant error, while a pair of error voltages of frequencies $u_c \pm \omega$ represents an alternating error of frequency ω.

Thus, if the line (or carrier) frequency u_c of the voltage that energizes the synchro repeaters is, for example, 60 cycles per sec., a constant input-output error produces an alternating error voltage of 60-cycles frequency. An alternately positive and negative error of 1, 2, or 3 cycles-per-sec. frequency produces, respectively, alternating voltages of 59 and 61 cycles, or 58 and 62 cycles, or 57 and 63 cycles-per-sec. frequency. If in a given servo system the error frequency may vary between, for example, zero and 3 cycles per sec., the error voltage frequencies will vary between 60 cycles and 60 ± 3 cycles, and may thus cover a frequency range of 57 to 63 cycles per sec. In other words, if the frequency band-width of the error is 3 cycles, the frequency bandwidth of the error voltage will be 6 cycles, *i.e.*, twice the error-frequency bandwidth.

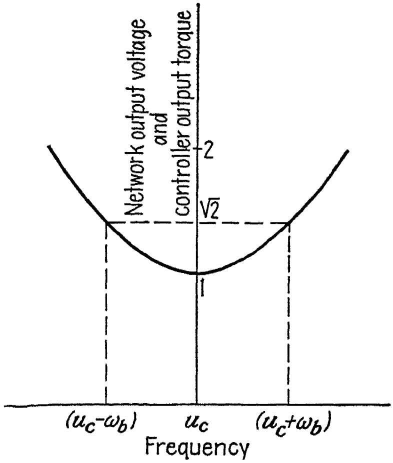

FIG. 7.6.—Controller voltage and output torque of a-c energized servomechanism.

Applying the results obtained from the discussion of Eqs. (7.7), (7.8), and (7.9) and curves of Fig. 7.1, the frequency characteristic of an error-rate damping or stabilizing network must then be such that, at the carrier frequency u_c, the network passes a voltage that will produce a torque K per unit error angle. At frequencies differing from the carrier frequency u_c by a modulation frequency ω, the network must pass a voltage producing a torque

$$K\sqrt{1 + \frac{\omega^2}{\omega_b{}^2}}$$

per unit error angle, leading the error angle by a phase angle equal to $\tan^{-1}(\omega/\omega_b)$.

Such a frequency response is shown graphically in Fig. 7.6. The carrier frequency u_c corresponds to zero error frequency. Frequencies of $u_c \pm \omega$ correspond to an error frequency ω. The curve of Fig. 7.6 is thus a transposition about the carrier frequency level u_c of the curve of Fig. 7.1. In view of the shape of the curve, an error-differentiating network with a characteristic such as that shown in Fig. 7.6 is sometimes referred to as a *notch* filter network.

A series-resonant circuit whose resistance is not zero, driven from

a high-impedance source, has the required characteristics. When such a circuit is placed in the error signal channel of a servo system, it will stabilize the mechanism in accordance with the preceding formulas of error-rate-damped systems.

Certain bridged-T and parallel-T resistance-capacitance networks possess similar characteristics. These networks are more easily built and used than the series-resonant circuit and will be analyzed below in greater detail.

Where modulated carrier voltages are employed in the following analysis, actual frequencies will be denoted by u. Hence

$$u = u_c \pm \omega \tag{7.30}$$

gives the relation between the carrier frequency u_c, the modulation frequency ω, and the two actual frequencies operating in the network.

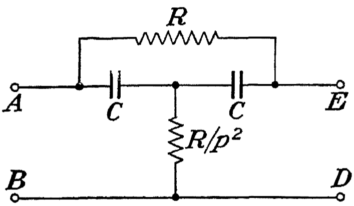

FIG. 7.7.—Bridged-T network.

Analysis of the Bridged-T Network.—A bridged-T network for use in error-rate stabilization in a servomechanism is represented schematically in Fig. 7.7. A network of this type has a frequency response similar to that of a series-resonant circuit with finite Q factor, which passes a minimum signal at some frequency, but never reaches a complete null with practical component values.

Equivalent Lattice Network.—For purposes of analysis, it is convenient to convert the bridged-T network of Fig. 7.7 into an equivalent lattice network. The transformation may be understood by first referring to Fig. 7.8, which shows a simple (unbridged) T network $GHMN$ and its equivalent lattice $ghmn$. It can be rigorously shown that these two networks have the same transfer characteristics. However, a simple demonstration of the equivalence of these two networks can readily be made in the limiting cases where the output terminals of the network are either open-circuited or short-circuited.

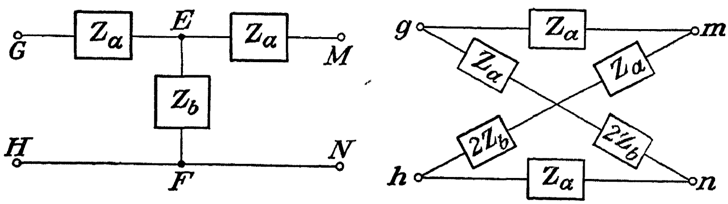

FIG. 7.8.—Simple T and equivalent lattice networks.

When the terminals M and N of the T network are open-circuited, the only circuit that is connected across the input terminals G and H

is the circuit $GEFH$. It consists of the two impedances Z_a and Z_b in series, so that the total impedance across GH is equal to the sum $Z_a + Z_b$.

Similarly, when the terminals m and n of the lattice are open-circuited, the circuit across terminals g and h consists of the circuit branch gmh connected in parallel with the circuit branch gnk. Each one of these two circuit branches has an impedance equal to $2Z_a + 2Z_b$. The over-all impedance across gh is thus equal to $Z_a + Z_b$ as in the case of the simple T network.

Conversely, when terminals M and N are short-circuited, the circuit connected across terminals G and H consists of the circuit branch GE in series with the parallel-connected branches EFH and $EMNH$. The total impedance across GH is then

$$Z_a + \frac{1}{1/Z_a + 1/Z_b} = \frac{Z_a^2 + 2Z_aZ_b}{Z_a + Z_b}. \tag{7.31}$$

In the same manner, when terminals m and n are short-circuited, the parallel-connected circuit branches gm and gn are in series with the parallel-connected branches hm and hn. The total impedance across terminals g and h is then

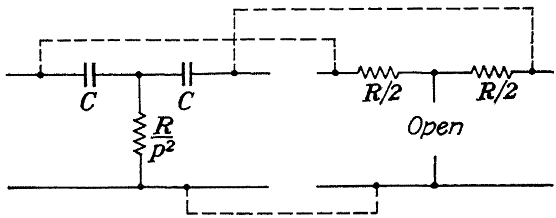

FIG. 7.9.—Transformation of bridged-T network.

$$\frac{1}{\dfrac{1}{Z_a} + \dfrac{1}{Z_a + 2Z_b}} + \frac{1}{\dfrac{1}{Z_a} + \dfrac{1}{Z_a + 2Z_b}} = \frac{Z_a^2 + 2Z_aZ_b}{Z_a + Z_b} \tag{7.32}$$

as in the preceding case.

In order now to convert the *bridged*-T network of Fig. 7.7 into an equivalent lattice, the network is considered as made up of two *simple* T networks in parallel—one composed of the two series capacitors C and shunting resistor R/p^2, and the other of two series resistors $R/2$ and an infinite shunting impedance (see Fig. 7.9). Transforming each of these two simple T networks in the manner shown in Fig. 7.8, and connecting the two equivalent lattices in parallel, results in the circuit represented in Fig. 7.10.

When a unit voltage is applied across terminals A and B, the voltages across terminals EB and DB are, respectively,

$$\begin{cases} e_{EB} = \dfrac{Z_{EB}}{Z_{EB} + Z_{AE}} & (7.33) \\ e_{DB} = \dfrac{Z_{DB}}{Z_{DB} + Z_{AD}} & (7.34) \end{cases}$$

where

$$Z_{EB} = Z_{AD} = \frac{2R}{p^2} + \frac{1}{juC} = \frac{2juRC + p^2}{jp^2uC}, \qquad (7.35,\ 7.36)$$

and

$$Z_{DB} = Z_{AE} = \frac{1}{juC + 2/R} = \frac{R}{2 + juRC}. \qquad (7.37)$$

The output voltage between D and E is then

$$e = e_{EB} - e_{DB} = \frac{Z_{EB} - Z_{DB}}{Z_{EB} + Z_{DB}}. \qquad (7.38)$$

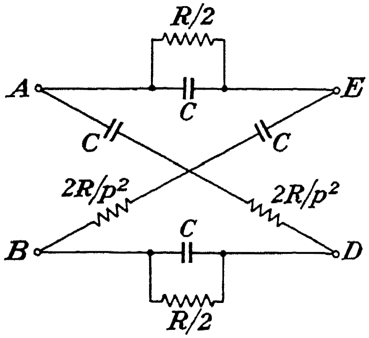

FIG. 7.10.—Lattice equivalent of bridged-T network.

Substituting Eqs. (7.36) and (7.37) in Eq. (7.38) gives

$$e = \frac{\dfrac{2juRC + p^2}{jp^2uC} - \dfrac{R}{2 + juRC}}{\dfrac{2juRC + p^2}{jp^2uC} + \dfrac{R}{2 + juRC}}. \qquad (7.39)$$

Multiplying the numerator and denominator by $-jp^2uC(2 + juRC)$ and collecting terms, this becomes

$$e = \frac{u^2R^2C^2 - p^2 - 2juRC}{u^2R^2C^2 - p^2 - juRC(p^2 + 2)}. \qquad (7.10)$$

Rationalizing, this expression is written

$$e = \frac{(u^2R^2C^2 - p^2)^2 + 2(p^2 + 2)u^2R^2C^2 + jp^2uRC(u^2R^2C^2 - p^2)}{(u^2R^2C^2 - p^2)^2 + (p^2 + 2)^2u^2R^2C^2}. \qquad (7.41)$$

This relation expresses the output voltage of the network as a function of the network constants R, C, p, and frequency u of the applied input sinusoidal voltage of unit amplitude. This voltage may be represented by a vector Oe, Fig. 7.11, the real and imaginary components of which are plotted at right angles to each other. As shown by the relation, Eq. (7.41), these two components are functions of the frequency u, and if the value of this frequency is varied, the end e of the voltage vector Oe describes a circle, as shown in the figure.

The output voltage e as expressed above has minimum amplitude for a frequency value

$$u_c = \frac{p}{RC} \qquad (7.42)$$

for which the imaginary component is zero. The output voltage is then in phase with the input voltage of the network.

Expression of Output Voltage as a Function of Relative-frequency Bandwidth.—In order to discuss Eq. (7.41) more easily, this equation may be simplified by expressing it in terms of a new variable defined as

$$x \equiv \frac{u}{u_c} - \frac{u_c}{u} \tag{7.43}$$

which is thus a function of the frequency u of the applied input voltage and of frequency u_c defined in Eq. (7.42).

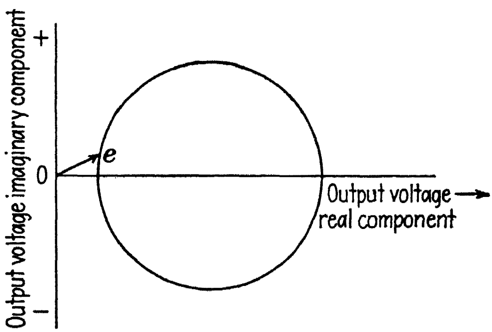

FIG. 7.11.—Output voltage of bridged-T network.

The Eq. (7.43) may be rewritten

$$x = \frac{u^2 - u_c^2}{uu_c} = \frac{(u + u_c)(u - u_c)}{uu_c} \tag{7.44}$$

In servo work, where the error frequency is small with respect to the carrier frequency, this last expression may be simplified by the permissible approximation that

$$u + u_c \cong 2u. \tag{7.45}$$

This, when substituted in Eq. (7.44), gives the expression

$$x = \frac{2(u - u_c)}{u_c} \tag{7.46}$$

or, from the relation, Eq. (7.30),

$$x = \frac{2\omega}{u_c} \tag{7.47}$$

where ω and u_c are, respectively, the error and carrier frequencies. Thus, x expresses the relative-frequency bandwidth of the error voltage (twice the error-frequency bandwidth) with respect to the frequency of the car-

rier voltage. Substituting Eq. (7.42) in Eq. (7.43) gives

$$x = \frac{uRC}{p} - \frac{p}{uRC} = \frac{u^2R^2C^2 - p^2}{puRC}, \tag{7.48}$$

and substituting this last expression in Eq. (7.41) after dividing the numerator and denominator by $(puRC)^2$, the output voltage of the network is finally expressed

$$e = \frac{x^2 + 2 + 4/p^2 + jpx}{x^2 + (p + 2/p)^2}. \tag{7.49}$$

Discussion of Network Equations.—The relation, Eq. (7.49) expresses the output voltage of the bridged-T network as a function of the network parameter p and the relative bandwidth x of the error voltage applied to the input terminals of the network.

When the error is constant (zero error frequency), $u = u_c$ or $x = 0$. The output voltage, as expressed above, is then equal to

$$e_c \doteq \frac{2}{p^2 + 2}. \tag{7.50}$$

On the other hand, when the error varies, its frequency is, in practice, sufficiently small to keep the values of x smaller than 0.5. Therefore, within the useful operating range for which the imaginary component of the output voltage is smaller than the real component, it may be seen from Fig. 7.11 that the real part of Eq. (7.49) can be considered constant and equal to the value e_c expressed in Eq. (7.50).

Since in practical circuits the value of p is generally equal to, or greater than 4, the term x^2 in the denominator of Eq. (7.49) is small in comparison to the term $[p + (2/p)]^2$ and can be neglected. The imaginary part of the expression of the output voltage Eq. (7.49) can then be written

$$e_j = \frac{px}{(p + 2/p)^2}. \tag{7.51}$$

Using Eq. (7.47), a value x_b can be defined as the value of the relative-frequency bandwidth x for which the error frequency ω assumes the value ω_b as given in Eq. (7.6).

$$x_b = \frac{2\omega_b}{u_c}. \tag{7.52}$$

From the introductory discussion of this chapter, it follows that the frequency corresponding to x_b is that frequency for which the real and imaginary components of the network output voltage are equal. In other words, it is that frequency for which the magnitude of the output voltage is $\sqrt{2}$ times as great as the output voltage corresponding to

$x = 0$ (zero error frequency). Equating the real and imaginary voltages, as given in Eqs. (7.50) and (7.51), and solving for x_b,

$$x_b = \frac{2}{p} + \frac{4}{p^3} \tag{7.53}$$

The last term in this expression being small in most practical cases, where p is greater than 4, this term may then be neglected, and an approximate value of x_b is

$$x_b = \frac{2}{p} \tag{7.54}$$

from which

$$p = \frac{2}{x_b}. \tag{7.55}$$

Substituting this last expression in Eq. (7.50), in order to determine the output voltage value when $x = 0$ in terms of the frequency at which the same circuit produces a voltage $\sqrt{2}$ times as great, gives the relation

$$e_c = \frac{x_b^2}{2 + x_b^2} \cong \frac{x_b^2}{2}. \tag{7.56}$$

Problem.—Following Eq. (7.18), an example was given to illustrate the effect on the performance of a servo system of adding error-rate damping to the previously

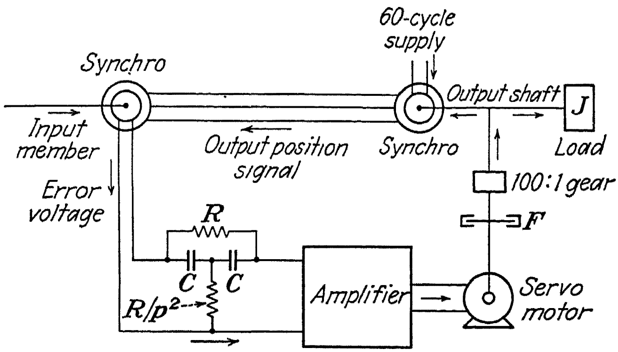

FIG. 7.12.—Servomechanism with viscous output damping and error-rate damping using a-c synchro repeater follow-up links.

existing viscous damping of the system. The actual design of this same system will now be considered. The conditions of the problem are recalled here for convenience.

A servo control system (Fig. 7.12) has inertia J and viscous output damping F that, referred to the motor shaft, have the values

$$J = 16 \times 10^{-6} \text{ slug-ft.}^2$$
$$F = 72 \times 10^{-6} \text{ ft.-lb. per radian per sec.}$$

The desired natural frequency of the system is

$$\begin{aligned}\omega_n &= 3.6 \text{ cycles per sec.}\\ &= (2\pi \times 3.6) \text{ or } 22\ 4 \text{ radians per sec.}\end{aligned}$$

From the oscillation curves given in a previous chapter, the damping ratio, in order that the oscillation may be damped out sufficiently fast, must be, for this system,

$$c = 0.4.$$

The system utilizes synchro repeater follow-up links energized from an a-c line of frequency

$$u_c = 60 \text{ cycles per sec.}$$
$$= (2\pi \times 60) \text{ or } 377 \text{ radians per sec.}$$

For small error angles, the synchro repeater supplies an error voltage of 1 volt per deg. input-output error angle (equivalent to a rate of 57 volts per radian error angle). The servo motor develops a torque of 0.1 ft.-lb. with an applied motor voltage of 100 volts, and its torque is proportional to voltage. The motor is geared down 100:1 to the load. Error-rate damping is obtained by inserting a bridged-T network in the error voltage circuit channel, utilizing capacitors of 0.01 mf. capacity. Calculate the other elements of the error-rate damping network and the required amplifier gain.

Solution: From Eqs. (7.18) and (7.17) it follows that

$$r = \frac{F}{2c\omega_n J} = \frac{72 \times 10^{-6}}{2 \times 0.4 \times 22.4 \times 16 \times 10^{-6}} = 0.25,$$

$$\omega_b = \frac{\omega_n}{2c(1 - r)} = \frac{22.4}{2 \times 0.4(1 - 0.25)} = 37.5 \text{ radians per sec.}$$
$$= 6 \text{ cycles per sec. (approximately).}$$

From Eq. (7.54),

$$x_b = \frac{2\omega_b}{u_c} = \frac{2 \times 37.5}{377} = 0.2$$

Then from Eq. (7.57),

$$p = \frac{2}{x_b} = \frac{2}{0.2} = 10.$$

Referring now to Eq. (7.42), and noting that the capcity C is chosen at the convenient size of 0.01 mf., the series resistance R of the bridged-T network is equal to

$$R = \frac{p}{Cu_c} = \frac{10}{10^{-8} \times 377} = 2{,}650{,}000 \text{ ohms,}$$

and the shunt resistor branch of the network is

$$\frac{R}{p^2} = 26{,}500 \text{ ohms.}$$

The output voltage of the network for an input voltage of 1 volt corresponding to a constant error of 1 deg. (zero error frequency, $x = 0$) is given by Eq. (7.56).

$$e_c = \frac{x_b^2}{2} = \frac{(0.2)^2}{2} = 0.02 \text{ volt.}$$

To calculate the amplifier gain, first determine the controller constant K from Eq. (7.10),

$$K = \omega_n^2 J = 22.4^2 \times 16 \times 10^{-6} = 8{,}100 \times 10^{-6} \text{ ft.-lb. per radian error angle.}$$

This torque is measured at the servo motor shaft, since the inertia J used in the calculation is referred to this shaft. The controller constant K_o referred to the output shaft is then

$$K_o = 100^2 \times 8{,}100 \times 10^{-6} = 81 \text{ ft.-lb. per radian error angle.}$$

On the other hand, this controller constant is, by definition, equal to the product.

$$\begin{aligned} K_o = &\text{ (synchro repeater volts per radian error)} \\ &\times \text{ (notch network output volts per input volts)} \\ &\times \text{ (amplifier gain)} \\ &\times \text{ (motor torque per applied motor volts)} \\ &\times \text{ (motor-to-load gear ratio).} \end{aligned}$$

Substituting the known values of these factors,

$$K_o = 81 = 57 \times 0.02 \times (\text{amplifier gain}) \times \frac{0.1}{100} \times 100$$

from which

$$\text{Amplifier gain} = \frac{81}{57 \times 0.02 \times 0.1} = 715.$$

It will be observed that the required amplifier gain is much greater than in the cases of viscous output damped servos studied before. The corresponding greater

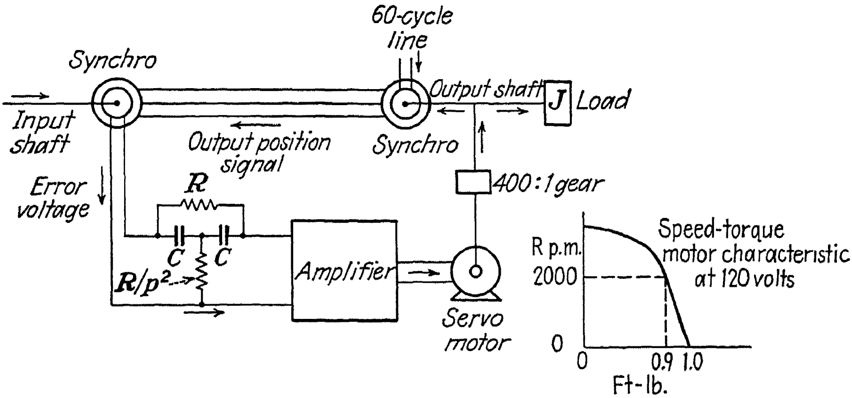

FIG. 7.13.—Servomechanism with viscous output damping and error-rate damping using a-c synchro repeater follow-up links.

complexity of the amplifier that must be used with this system employing error-rate damping represents a greater cost of the equipment. This is the price of the improved correction of the input-output error obtainable with this type of servomechanism.

Problem.—A second example of error-rate-damped servo calculation is given below. In the servo system shown in Fig. 7.13 the inertia, referred to the motor shaft, is

$$J_M = 200 \times 10^{-6} \text{ slug-ft.}^2$$

Friction in the system is negligible, but viscous damping is obtained from the motor speed-torque characteristics, which are as shown in the figure and correspond over the useful operating range of the motor to a torque reduction of 10 per cent at a motor speed of 2,000 r.p.m. The motor torque is proportional to the applied motor voltage. The motor shaft is geared down to the output shaft by a 400:1 ratio gear box. A synchro repeater follow-up link provides a 60-cycle error voltage at the rate of 57 volts per radian error angle (for small values of this angle). The desired damping ratio c is 0.25. Error-rate damping is introduced by means of a bridged-T network having capacitors of 0.01 mf. capacity each. Determine the other net-

work elements, and the amplifier gain required, knowing that the steady-state error must not exceed an angle of 15 min. at a speed ω_1 of 5 r.p.m.

Solution: The controller constant K and natural frequency ω_n are first calculated from the relations

$$\begin{cases} F\omega_1 = K\theta \\ \omega_n = \sqrt{\dfrac{K}{J}} \end{cases}$$

in which F, ω_1, θ, and J are given or can be calculated from the specified characteristics of the system. Thus

$$\theta = 15 \text{ min.} = 0.25 \text{ deg.} = \frac{0.25}{57.3} = 0.00436 \text{ radian}$$

$$\omega_1 = 5 \text{ r.p.m.} = \frac{5 \times 2\pi}{60} \text{ or } 0.523 \text{ radian per sec.}$$

$$J_o = 200 \times 10^{-6} \times 400^2 = 32 \text{ slug-ft.}^2 \text{ at output shaft}$$

$$F \text{ (from motor curves)} = \frac{\Delta T}{\Delta S} = \frac{0.1 \text{ ft.-lb.}}{200 \text{ radian per sec.}} = 500 \times 10^{-6}$$

and

$$F_o \text{ (at output shaft)} = 500 \times 10^{-6} \times 400^2 = 80 \text{ ft.-lb. per radian per sec.}$$

It then follows that the controller constant in ft.-lb. per radian is

$$K = \frac{F\omega_1}{\theta} = \frac{80 \times 0.523}{0.00436} = 9{,}600 \text{ (referred to output shaft)}$$

and

$$\omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{9{,}600}{32}} = 17.3 \text{ radians per sec.}$$

To calculate the notch network constants, use the same relations as in preceding example.

$$r = \frac{F}{2c\omega_n J} = \frac{80}{2 \times 0.25 \times 17.3 \times 32} = 0.3;$$

$$\omega_b = \frac{\omega_n}{2c(1-r)} = \frac{17.3}{2 \times 0.25(1-0.3)} = 50;$$

$$x_b = \frac{2\omega_b}{u_c} = \frac{2 \times 50}{2\pi \times 60} = 0.267;$$

$$p = \frac{2}{x_b} = \frac{2}{0\ 267} = 7.5;$$

$$RC = \frac{p}{u_c} \quad \text{from which} \quad R = \frac{p}{Cu_c} = \frac{7.5}{0.1 \times 10^{-6} \times 377} = 2 \text{ megohms};$$

and

$$\frac{R}{p^2} = \frac{2{,}000{,}000}{7.5 \times 7.5} = 36{,}000 \text{ ohms.}$$

For a constant error alternating voltage of 1 volt, the output voltage of the notch network is

$$e_c = \frac{x_b^2}{2} = \frac{0.267^2}{2} = 0.035 \text{ volt.}$$

Calculating the amplifier gain as in the preceding exercise,

$$K = 9{,}600 = 57 \times 0.035 \times \text{gain} \times 1/20 \times 400$$

from which

$$\text{Amplifier gain} = \frac{9{,}600 \times 120}{57 \times 0.035 \times 400} = 1{,}480.$$

Problem.—A servomechanism has a natural frequency of 20 radians per sec. and a damping constant c of 0.3. One-fourth of the damping present in the system is the result of viscous output damping, the remainder being due to error-rate damping produced by a bridged-T notch circuit. If a 60-cycle-per-sec. synchro repeater signal is used, determine the relative frequency bandwidth x_b the notch circuit must possess. (*Ans.:* $x_b = 0.23$.)

Problem.—A servomechanism requires a controller constant of 10,000 ft.-lb. at the output shaft per radian error. The controller comprises a synchro error circuit, a notch stabilizer, an amplifier, an induction motor, and a gear box. The synchro error circuit produces 1 volt per deg. error. The notch stabilizer consists of a bridged-T network, and the values of the components are such that x_b equals 0.2. With one phase constantly excited, the motor produces 1 ft.-lb. locked torque when a voltage of 115 volts is applied by the amplifier across the other phase of the motor. The motor output shaft is geared down 400:1 to the output shaft of the servo system. Determine the required gain of the amplifier. (Answer: 2,500.)

Analysis of the Parallel-T Null Network.—Another type of network suitable for use in error-rate stabilization of a servomechanism is the parallel-T null network[1] shown schematically in Fig. 7.14.

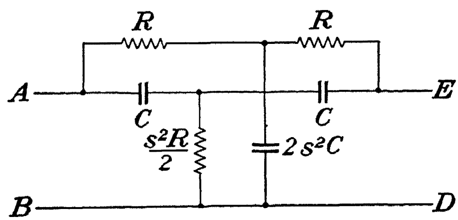

FIG. 7.14.—Parallel-T network.

The circuit illustrated here has a frequency response which differs from that of a series resonant circuit whose resistance is not zero, and of the bridged-T network studied in the preceding paragraphs, in that complete attenuation of the input signal at carrier frequency is easily approached with practical components. Since, as shown in Fig. 7.6, the network inserted in the error signal channel of the servo system must pass a finite signal at the carrier frequency and must accentuate the signal at frequencies differing from that of the carrier, it becomes necessary to modify the parallel-T network in order to adapt it for servo work. Figure 7.15 illustrates a modification that allows a portion of the carrier voltage to pass around the network.

The following analysis first deals with the parallel-T circuit itself. The concept of a modifying signal will be considered later.

[1] This circuit was first described by Augustadt in U.S. Patent 2,106,785 for use in power supply filter circuits.

Equivalent Lattice Network.—As in the case of the bridged-T network, it is simpler to analyze an equivalent lattice network instead of the parallel-T network itself. Conversion of the circuit of Fig. 7.14 is accomplished by considering this circuit as made up of two simple T networks connected in parallel and by applying to each of these the transformation illustrated before in Fig. 7.8. The equivalent lattice network of Fig. 7.16 is then obtained.

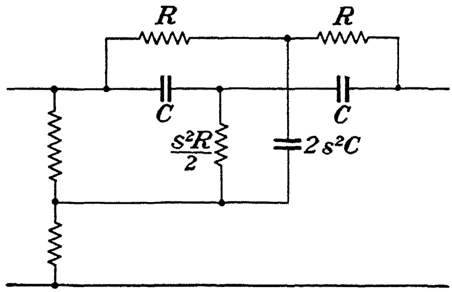

FIG. 7.15.—Parallel-T network with by-pass channel.

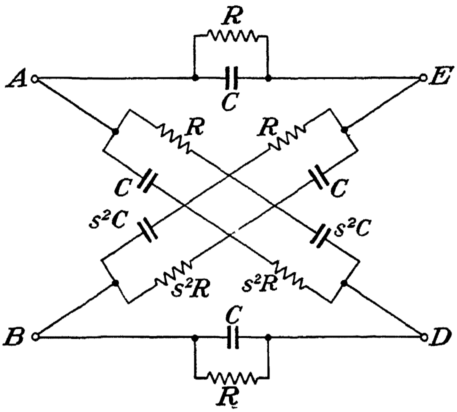

FIG. 7.16.—Lattice equivalent of parallel-T network.

When a unit voltage is applied across the input terminals A and B, the output voltage across terminals D and E is, according to the previously derived equation (7.38),

$$e = \frac{Z_{EB} - Z_{DB}}{Z_{EB} + Z_{DB}}, \tag{7.38}$$

where

$$Z_{EB} = Z_{AD} = \frac{1}{\dfrac{1}{R + \dfrac{1}{jus^2C}} + \dfrac{1}{s^2R + \dfrac{1}{juC}}}; \tag{7.57}$$

or

$$Z_{EB} = Z_{AD} = \frac{jus^2RC + 1}{juC(s^2 + 1)}, \tag{7.58}$$

and

$$Z_{DB} = Z_{AE} = \frac{1}{juC + \frac{1}{R}} = \frac{R}{1 + juRC}. \tag{7.59}$$

Substituting Eqs. (7.58) and (7.59) in Eq. (7.38), the output voltage is expressed

$$e = \frac{\frac{jus^2RC + 1}{juC(s^2 + 1)} - \frac{R}{1 + juRC}}{\frac{jus^2RC + 1}{juC(s^2 + 1)} + \frac{R}{1 + juRC}}. \tag{7.60}$$

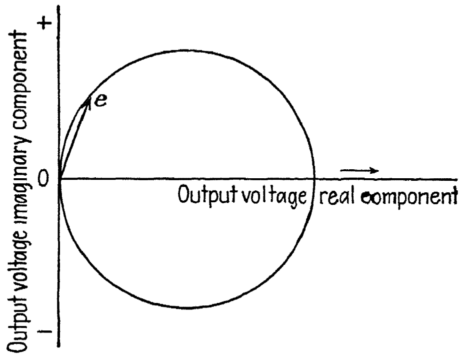

FIG. 7.17.—Output voltage of parallel-T network.

Expanding and collecting terms,

$$e = \frac{u^2s^2R^2C^2 - 1}{u^2s^2R^2C^2 - 1 - 2juRC(s^2 + 1)}. \tag{7.61}$$

Rationalizing, the expression becomes

$$e = \frac{(u^2s^2R^2C^2 - 1)^2 + 2juRC(u^2s^2R^2C^2 - 1)(s^2 + 1)}{(u^2s^2R^2C^2 - 1)^2 + 4u^2R^2C^2(s^2 + 1)^2}. \tag{7.62}$$

Inspection of this expression shows that, irrespective of the value of u, the denominator always has a finite value, while the numerator, and therefore also the output voltage of the network, becomes zero for a frequency value u_c.

$$u_c = \frac{1}{sRC}. \tag{7.63}$$

Equation (7.62) represents the output voltage e of the network as a function of the circuit parameters R, C, s, and the frequency u of the applied sinusoidal voltage of unit amplitude. This output voltage may

be represented by a vector Oe, Fig. 7.17, by plotting the real and the imaginary voltage components, as given in Eq. (7.62), along rectangular axes. As shown by the equation, both components are functions of the frequency u, and if the value of this frequency is varied, the end e of the voltage vector describes a circle tangent to the imaginary axis at the point of origin of the coordinates.

Expression of Output Voltage as a Function of Relative-frequency Bandwidth, and Discussion of the Equation.—Discussion of the Eq. (7.62) is simplified by introducing a new frequency variable x defined as before.

$$x = \frac{u}{u_c} - \frac{u_c}{u} \tag{7.43}$$

which, after substituting Eq. (7.63), becomes

$$x = usRC - \frac{1}{usRC}, \tag{7.64}$$

or

$$x = \frac{u^2s^2R^2C^2 - 1}{usRC}. \tag{7.65}$$

Writing this expression in Eq. (7.62) after dividing the numerator and denominator by $u^2s^2R^2C^2$, the output voltage equation becomes

$$e = \frac{x^2 + 2j(s + 1/s)x}{x^2 + 4(s + 1/s)^2}. \tag{7.66}$$

As previously noted, the magnitude of x is generally less than 0.5 under operating conditions generally encountered in practice. The quantity x^2 in Eq. (7.66) can then be neglected, and the output voltage is expressed, with sufficient accuracy,

$$e = \frac{jx}{2(s + 1/s)}. \tag{7.67}$$

Thus, as illustrated in Fig. 7.17 and as shown by the relation, Eq. (7.67), the output voltage of the network for small values of x is substantially in phase quadrature with the applied input voltage (error voltage), and is approximately proportional to x.

The amount e_c of error voltage, of carrier frequency, which must be by-passed around the network, in the manner described in relation to Fig. 7.15, is then readily determined in recalling that at the frequency corresponding to the value x_b of x the imaginary component of the output voltage, as given by Eq. (7.67), must have the same magnitude as the real component voltage applied to the controller. Thus

$$e_c = \frac{x_b}{2(s + 1/s)}. \tag{7.68}$$

This derivation is made under the valid assumption that x_b being small, the contribution of the in-phase (real) voltage from the parallel-T network itself is negligible.

The value of s for which minimum attenuation is obtained for a given bandwidth is equal to unity. The following table gives the value of the circuit constants for various values of s.

s	$\sqrt{2}$	1	$1/\sqrt{2}$
Series R	R	R	R
Shunt R	R	$R/2$	$R/4$
Series C	C	C	C
Shunt C	$4C$	$2C$	C
Attenuation	$x_b/4.24$	$x_b/4$	$x_b/4.24$
u_c	$1/\sqrt{2}\,RC$	$1/RC$	$\sqrt{2}/RC$

The preceding discussion may be summarized by saying that the parallel-T network feeds into the controller a voltage which is in quadrature with the error signal voltage, and which varies with the error frequency at a rate

$$\frac{de_i}{dx} = \frac{1}{2[s + (1/s)]}. \tag{7.69}$$

In addition to this voltage, a voltage in phase with the error signal voltage must be supplied to the controller. Its magnitude must be equal to that of the quadrature voltage passed by the parallel-T network at the frequency corresponding to x_b as defined.

It should be noted that for a given natural frequency ω_n of the servo system the error-frequency bandwidth ω_b is determined by the desired damping ratio c.

$$\omega_b = \frac{\omega_n}{2c(1 - r)}.$$

Not only does this make the damping network more critical to frequency, by narrowing the bandwidth as the damping factor c is made larger, but it also increases the gain required from the amplifier, since it reduces the voltage passed at the frequency corresponding to x_b. This is illustrated in the following table, calculated for a carrier frequency of 60 cycles per sec. and for a parallel-T network in which s is equal to unity.

ω_b cycles per sec.	$x_b = 2\omega_b/u_c$	$x_b/4$ = volts output with 1 volt input at carrier frequency
3	$1/10$	$1/40$ volt
6	$2/10$	$1/20$ volt
9	$3/10$	$3/40$ volt
12	$4/10$	$1/10$ volt

CHAPTER VIII

ANALYSIS OF SERVOMECHANISMS WITH INTEGRAL CONTROL

The primary purpose of introducing viscous output damping or error-rate damping in a servomechanism, as described in the preceding chapters, is to reduce the amplitude and duration of the transient error that occurs whenever the system is disturbed from its prevalent operating condition. This damping or stabilization is obtained at the cost of producing a steady-state error in the case of viscous output damping, or of requiring added amplifier gain in the case of error-rate damping, with associated increased natural frequency, which may be objectionable in some cases.

These stabilization methods are effective when the energy-consuming load is small. Error-rate damping is particularly effective in servos having a large inertia load, where its main purpose is to improve the transient response without introducing a steady-state error. However, these stabilization methods do not prevent the input-output error from increasing when the amount of power drawn from the system by the driven load is increased.

Methods of integral control, to be studied in this chapter, are generally used in addition to the aforementioned stabilization methods to reduce the error of the system without appreciably raising the natural frequency, while at the same time increasing the effective controller gain and torque. Integral control is therefore particularly valuable in cases of heavy external load demands, such as are encountered in numerous industrial applications.

Mechanical Analogy.—As in the case of servo systems discussed previously, a mechanical analogy may be helpful in understanding the mode of operation of integral control. As in previous explanations, the servomechanism will be likened to a pendulum, since both devices possess inertia and elasticity, or the equivalent.

It was shown in Chap. V that a simple servomechanism with viscous output damping can be compared to a pendulum, the pivot of which is being displaced horizontally, while a vane attached to the pendulum mass is dragged through a damping fluid contained in a stationary tank for the purpose of damping out the oscillations and stabilizing the system. This was illustrated in Fig. 5.3, which is reproduced here for convenience as Fig. 8.1. The response of the pendulum mass to the motion of the pivot was found to be identical with that of a servomechanism, the input and output member positions of which are represented, respectively, by

the horizontal positions of the pivot and the mass of the pendulum. When the carriage C is driven at constant speed along the rail RR', the pendulum assumes in the steady-state the slanting position shown in the sketch. This is due to the friction between the vane and the damping fluid. A discrepancy or error E thus arises between the instantaneous positions of the pendulum mass M and the pivot O. The error is proportional to the friction, and so constant-speed motion of the system requires a constant driving force to be applied, equal and opposite to the friction, hence proportional to the error.

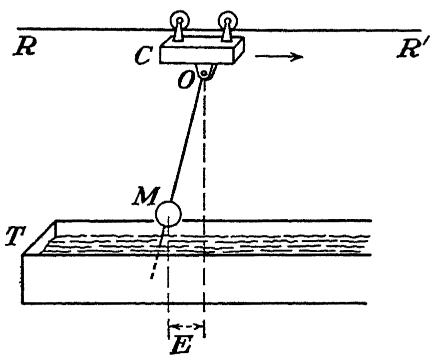

FIG. 8.1.—Pendulum analogy of viscous-damped servomechanism.

Suppose, as shown in Fig. 8.2, that the carriage C is filled with sand and that its floor is provided with two small funnels H and K, through which the sand may escape when either lower orifice is open. The pendulum operates a slide S, which closes the openings of both funnels when the pendulum is in a vertical position, but opens either the forward funnel K or the rear funnel H, depending on whether the pendulum is displaced forward or backward of the vertical position. Attached to the pendulum are also two cups D and F into which the sand escaping through H or K, respectively, may fall. As in the case of Fig. 8.1, a vane is fixed to the pendulum mass and dips in the damping fluid contained in the stationary tank T.

Let now the carriage C be driven at constant speed from left to right along the rail RR' against the friction of the vane in the damping fluid. As in the simple viscous-damped system of Fig. 8.1, the pendulum will tend to assume a backward slanting position, as shown in the sketch A of Fig. 8.2, producing a position error E between the mass M and pivot. This, however, displaces the slide S and opens the rear funnel H, through which sand then escapes from the carriage C into the cup D. The resulting unbalance of the pendulum tends to bring it into a vertical position, as shown in the sketch B of the figure. As the pendulum approaches the *zero-error*, or vertical, position, the slide S gradually closes the funnel H and finally cuts off the flow of sand when the pendulum reaches the vertical position. A similar error-canceling action would take place, through the forward funnel K, if the pendulum should swing forward of the vertical position.

It may be assumed that the opening of the funnel H and the flow

of sand into the cup D are proportional to the instantaneous deviation or error E of the pendulum from the vertical position. The weight of sand collected by the cup during the period of time required for the error to be reduced to zero and for the pendulum to be brought to the vertical position can then readily be calculated. Thus, consider this period of time to be made up of a succession of elementary time intervals Δt so short that the funnel opening may be looked upon as remaining constant during any one of them. If a_1 is the area of the opening during one such time interval Δt_1, the weight of sand w_1 flowing out of the funnel during this time interval is proportional to the product $a_1 \Delta t_1$. Since the opening area a_1 is proportional to the corresponding error value e_1, the weight w_1 is also proportional to the product $e_1 \Delta t_1$, or equal to this product if a suitable unit of weight is used.

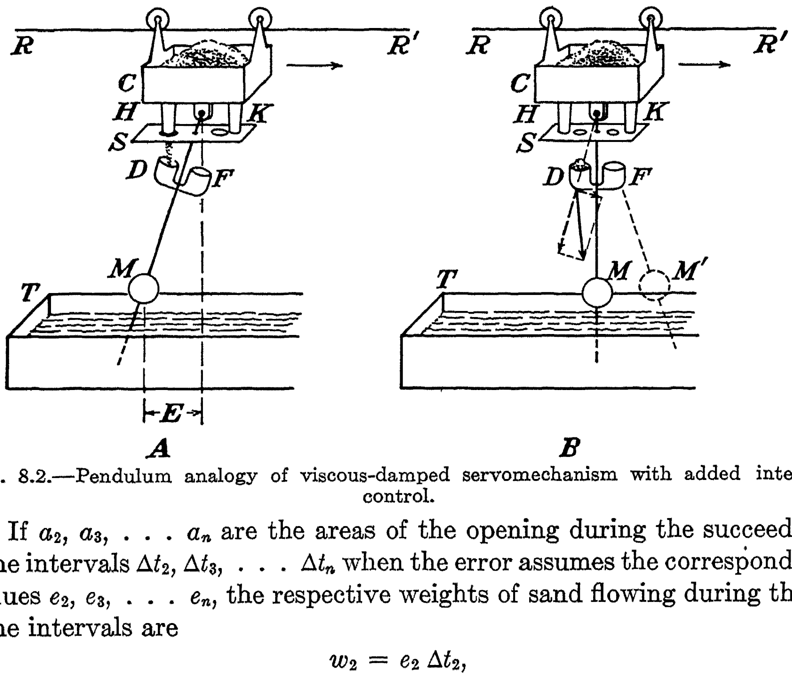

FIG. 8.2.—Pendulum analogy of viscous-damped servomechanism with added integral control.

If $a_2, a_3, \ldots a_n$ are the areas of the opening during the succeeding time intervals $\Delta t_2, \Delta t_3, \ldots \Delta t_n$ when the error assumes the corresponding values $e_2, e_3, \ldots e_n$, the respective weights of sand flowing during these time intervals are

$$\begin{aligned} w_2 &= e_2 \, \Delta t_2, \\ w_3 &= e_3 \, \Delta t_3, \\ &\cdots \\ w_n &= e_n \, \Delta t_n. \end{aligned}$$

The total weight of sand collected by the cup is then

$$\begin{aligned} w &= w_1 + w_2 + w_3 + \cdots + w_n \\ &= e_1 \, \Delta t_1 + e_2 \, \Delta t_2 + \cdots + e_n \, \Delta t_n \\ &= \sum_{n = 1, 2, \cdots} e_n \, \Delta t_n. \end{aligned}$$

The assumption that the funnel opening remains constant during any elementary time interval is, of course, more closely justified as these time intervals are made smaller. If they are made infinitesimally small and infinitely numerous, the preceding summation becomes equal to the *time integral of the error.*

$$w = \int e\,dt;$$

hence the name of *integral control* given to this method of correcting the input-output error of the system to zero.

There is a fundamental difference between the operating conditions of the servos previously studied and those of a servo employing integral control: while the driving force (or torque, in case of rotary motion) in the steady-state condition is proportional to the error in these earlier servo systems and becomes equal to zero with the error, a finite driving force may be operating in an integral control system, even when the error is zero.[1]

Another characteristic feature of a servo employing integral control is that the system will not come to rest, when the input member is stopped, until the time integral of the error has become equal to zero. This may be illustrated by referring again to the pendulum analogy pictured in Fig. 8.2. As explained in relation to the diagram B of that figure, the pendulum is subjected to the retarding friction drag between the pendulum vane and the damping liquid, caused by the translatory motion of the system toward the right. This drag tends to give to the pendulum an oblique position similar to that shown in Fig. 8.1. However, the pendulum is maintained in vertical position because it is unbalanced by the weight of the sand, which has accumulated in the cup D during the transient period.

When the system is stopped, the friction drag disappears, but the unbalance remains, which tends to give to the pendulum a forward slanting position, shown as M' in dotted line. This, in turn, opens the funnel K, allowing sand to flow from the carriage C into the cup F until a weight of sand equal to that contained in the cup D restores the balance of the pendulum. The pendulum then assumes its normal vertical position of rest.

In equivalent manner the controller motor of a servo will have to develop a negative torque, *i.e.*, a torque directed opposite to the driving torque, when the system is being stopped. This action is independent of such *overswing* or oscillation as may normally occur, as a result of the inertia of the system, when the input member is stopped.

[1] The same zeroing effect is obtained, with integral control, when an energy-consuming external load is connected to the output member. In the above described pendulum analogy, the steady-state error will be zero even when the pendulum is performing work, such as lifting some weight attached to it, for example.

As described above, integral control is applied to servomechanisms with viscous output damping for neutralizing the steady-state error resulting from the output friction drag. It can be applied as well to servomechanisms having both viscous and error-rate damping. It is this general case which will be analyzed mathematically in the paragraphs that follow, after which practical means for obtaining integral control will be described and discussed briefly.

Equation of Servo System with Integral Control.—A servomechanism with integral control is shown schematically in Fig. 8.3. This system uses both viscous output damping and error-rate damping. As in the previous cases the output member is connected to a load possessing inertia. The position of the output member is subtracted from that of the input member by a differential device, and the resulting difference, or error, actuates a controller. Viscous output damping, producing a retarding force proportional to the output speed, is present in the system

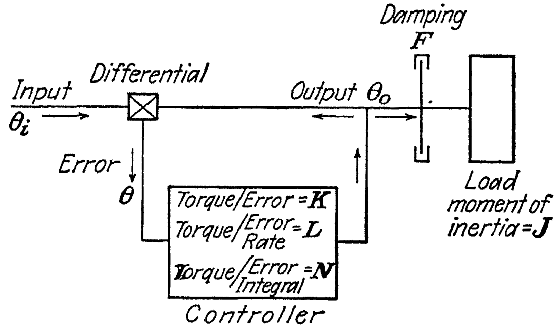

FIG. 8.3.—Servomechanism with viscous output damping, error-rate damping, and integral control.

due to friction in the moving parts or to a speed-torque motor characteristic curve with negative slope. Error-rate damping is also assumed to be present, producing an accelerating force proportional to the time rate of change of the error. Finally, the controller is arranged to produce an additional torque proportional to the time integral of the error, *i.e.*, to the summation of the product of error and incremental units of time. In addition to the symbols used in the previous chapters, the symbol N will be used here to represent the constant of proportionality between torque and error integration.

In the system just described, as in the other systems previously discussed, the sum of the forces must be equal to zero. In other words, the accelerating forces must equal the retarding forces.

The accelerating forces, as produced by the controller, comprise

1. A torque $K\theta$ proportional to the error.

2. A torque $L\,d\theta/dt$ proportional to the time rate of change of the error.

3. A torque $N\int\theta\,dt$ proportional to the time integral of the error.

On the other hand, the retarding forces are

1. The inertia torque $J\,d^2\theta_o/dt^2$ proportional to the output acceleration.

2. The viscous drag $F\,d\theta_o/dt$ proportional to the output speed.

The equation of motion of the system is thus written

$$K\theta + L\frac{d\theta}{dt} + N\int\theta\,dt = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt}. \tag{8.1}$$

Since the error is, by definition, the difference between the input and output positions,

$$\theta = \theta_i - \theta_o, \tag{8.2}$$

the equation may be rewritten in terms of the input angle and error angle by substituting Eq. (8.2) in Eq. (8.1),

$$J\frac{d^2\theta}{dt^2} + (L + F)\frac{d\theta}{dt} + K\theta + N\int\theta\,dt = J\frac{d^2\theta_i}{dt^2} + F\frac{d\theta_i}{dt}. \tag{8.3}$$

Step Input Function.—As in the preceding chapters and in order to allow a comparison of the transient response of the present system with that of the systems previously studied, a step input function will be applied to the servomechanism. That is to say, the input member is considered as motionless up to a time $t = 0$ when it is suddenly set in motion at some constant speed ω_1. The input angle θ_i thus increases linearly with time.

$$\begin{cases} \theta_i = 0, & t < 0 \qquad (8.4) \\ \theta_i = \omega_1 t, & t \geqq 0. \qquad (8.5) \end{cases}$$

These conditions were represented graphically in Chap. IV, Fig. 4.2.

In order to find the differential equation of the system with this type of input function, after the starting instant of the input member, the actual values of θ_i and its successive time derivatives are substituted in Eq. (8.3). According to Eq. (8.5), these derivatives are

$$\frac{d\theta_i}{dt} = \omega_1 \tag{8.6}$$

$$\frac{d^2\theta_i}{dt^2} = 0. \tag{8.7}$$

Equation (8.3) then becomes

$$J\frac{d^2\theta}{dt^2} + (L + F)\frac{d\theta}{dt} + K\theta + N\int\theta\,dt = F\omega_1, \tag{8.8}$$

which is the equation of motion of the error of the system.

Steady-state Error.—Inspection of Eq. (8.8) shows that if the system is stable and sufficient time has elapsed for the transient initiated by the step input function to die out, no steady-state error can exist in the system. For if any such error should exist, the accelerating torque represented by the integral term of the equation would grow indefinitely with time; this is incompatible with the fact that the second member of the equation is constant. The form of the transient must be such that integration of the transient error produces controller torque to compensate for the retarding viscous output torque.

General Transient Solution.—Solution of the problem of finding the error θ of a servomechanism with integral control requires that the general transient solution first be found. This general solution is then adapted to the particular system by determining its arbitrary constants from the boundary conditions of the system.

The generalized solution of the transient error depends only on the servo system itself and is not affected by the form of the input function. A valid solution can therefore be found by setting, for convenience, the input function of Eq. (8.3) equal to zero,

$$J\frac{d^2\theta}{dt^2} + (L + F)\frac{d\theta}{dt} + K\theta + N\int \theta\, dt = 0 \tag{8.9}$$

which, divided through by J, becomes

$$\frac{d^2\theta}{dt^2} + \frac{(L + F)}{J}\frac{d\theta}{dt} + \frac{K}{J}\theta + \frac{N}{J}\int \theta\, dt = 0. \tag{8.10}$$

When $N = 0$, or in other words when there is no integral control, this equation is the same as Eq. (6.10) of Chap. VI. In the course of that chapter it was found that the natural frequency ω_n of the servo is

$$\omega_n = \sqrt{\frac{K}{J}}. \tag{8.11}$$

Also, the damping ratio c was defined as the ratio of the damping $(L + F)$ actually present to the damping $(L + F)_c$ required to damp the system critically.

$$c = \frac{(L + F)}{(L + F)_c} \tag{8.12}$$

Finally, it was found that critical damping in a system having no integral control is obtained when

$$(L + F)_c = 2\sqrt{KJ}. \tag{8.13}$$

Substituting these three relations, Eqs. (8.11), (8.12), and (8.13) in Eq. (8.10) gives

$$\frac{d^2\theta}{dt^2} + 2c\omega_n\frac{d\theta}{dt} + \omega_n^2\theta + \frac{N}{J}\int \theta\, dt = 0. \tag{8.14}$$

From dimensional considerations it can be seen that the factor N/J of the last term must have the dimensions of the third power of frequency. A new constant s is therefore introduced, defined as the *constant of proportionality* between this factor and the cube of the natural frequency,

$$\frac{N}{J} = s\omega_n^3. \tag{8.15}$$

Substituting this relation in Eq. (8.14), there results

$$\frac{d^2\theta}{dt^2} + 2c\omega_n \frac{d\theta}{dt} + \omega_n^2\theta + s\omega_n^3 \int \theta \, dt = 0. \tag{8.16}$$

Differentiating this last equation gives

$$\frac{d^3\theta}{dt^3} + 2c\omega_n \frac{d^2\theta}{dt^2} + \omega_n^2 \frac{d\theta}{dt} + s\omega_n^3\theta = 0. \tag{8.17}$$

Assuming, as in the cases studied in the preceding chapters, that the error is of the form

$$\theta = A\epsilon^{pt} \tag{8.18}$$

and substituting this expression and its derivatives in Eq. (8.17) permits the value of the constant p to be determined.

The first, second, and third time derivatives of Eq. (8.18) are

$$\frac{d\theta}{dt} = pA\epsilon^{pt} \tag{8.19}$$

$$\frac{d^2\theta}{dt^2} = p^2A\epsilon^{pt} \tag{8.20}$$

$$\frac{d^3\theta}{dt^3} = p^3A\epsilon^{pt}. \tag{8.21}$$

Writing Eqs. (8.18 to 8.21) in Eq. (8.17), this becomes

$$p^3A\epsilon^{pt} + 2c\omega_np^2A\epsilon^{pt} + \omega_n^2pA\epsilon^{pt} + s\omega_n^3A\epsilon^{pt} = 0 \tag{8.22}$$

or, after dividing by $A\epsilon^{pt}$,

$$p^3 + 2c\omega_np^2 + \omega_n^2p + s\omega_n^3 = 0. \tag{8.23}$$

This cubic equation in p is expressed in terms of ω_n and c, which are, respectively, the natural frequency and the damping constant of a servo without integral control, and s the integral-control constant.

When s is zero, *i.e.*, when no integral control is used in the system, the last term of the left-hand member of the equation disappears, and the equation (after dividing through by p) reduces to the quadratic equation

$$p^2 + 2c\omega_np + \omega_n^2 = 0. \tag{8.23a}$$

This is the same as the corresponding equation of the previous chapters [for example, Eq. (4.15) can be written in the form of Eq. (8.23*a*) by dividing through by J and then applying the relations, Eqs. (4.50) and (4.58)]. Thus when $s = 0$, the solution will be the same as was encountered in the systems previously discussed. The values of p obtained from Eq. (8.23*a*) are expressed in terms of the system constants c and ω_n, and when these values are complex expressions with real and imaginary parts (as is the case for most practical servo applications), the error θ is a damped oscillation of frequency $\omega_n \sqrt{1 - c^2}$ and damping rate $c\omega_n$.

When integral control is introduced in the system, the constant s differs from zero, and the values for the exponential factor p in the error Eq. (8.18) must be calculated from Eq. (8.23). In addition to the system constants c and ω_n, the integral control constant s then enters into the expression of the factor p. Since this factor, p, determines the time-dependence characteristics of the error θ, these characteristics will be altered accordingly. As will be found below, both the oscillation frequency and the damping constant are changed, as though the system had a natural frequency and damping ratio different from the original ones. A simple exponentially decaying term is also added in the expression of the transient error, which is not present in the servo systems studied in the previous chapters.

The solution of the cubic equation (8.23) generally leads to three roots. However, the expressions for these roots in terms of the equation coefficients and system parameters are rather complicated[1] and do not lend themselves readily to a discussion of the various conditions encountered. It is therefore more illustrative to obtain the solution in terms of the actual resonant frequency and damping constant of the oscillatory component, rather than carry the original terms ω_n, c, s into the solution. For this purpose[2] a cubic equation may be set up in terms of the actual (modified) resonant frequency and damping ratio, to be denoted by ω_g and g, whose solution can be obtained simply by comparison with the results of previous chapters. Then, by equating corresponding coeffi-

[1] See, for example, J. M. Rice in O. W. Eshbach, "Handbook of Engineering Fundamentals," pp. 2-13, 2–14, John Wiley & Sons, Inc., New York, 1936; or I. S. Sokolnikoff and E. S. Sokolnikoff, "Higher Mathematics for Engineers and Physicists," pp. 86–91, McGraw-Hill Book Company, Inc., New York, 1941.

[2] Methods and charts for the solution of cubic equations were established by E. Jahnke and F. Emde, "Tables of Functions," pp. 20–30, B. G. Teubner, Leipzig, 1933, and pp. 20–30 (Addenda), Dover Publications, New York, 1943; by H. K. Weiss, Constant Speed Control, *J. Aeronaut. Sci.*, February, 1943; Y. J. Liu, Servomechanisms, Charts for Verifying Their Stability and for Finding the Roots of Their Third and Fourth Degree Characteristic Equations, Massachusetts Institute of Technology, October, 1941; Leroy W. Evans, Solution of the Cubic Equation and the Cubic Charts, Massachusetts Institute of Technology, 1943.

cients, it is possible to determine the relations between the new parameters ω_g and g and the original constants ω_n, c, and s.

The new equation, like any other cubic equation with real coefficients, can be written as the product of two polynomials of the first and second degrees, respectively,

$$(p + m)(p^2 + np + k) = 0 \tag{8.24}$$

where the constants m, n, and k are real quantities.

The equation thus has at least one real root,

$$p = -m, \tag{8.25}$$

for which the first factor in Eq. (8.24) becomes equal to zero. When this value of p is written in Eq. (8.18) it is seen to contribute a simple exponentially decaying term

$$\theta = A\epsilon^{-mt} \tag{8.26}$$

to the error of the system considered.

In order to utilize the results obtained in previous chapters to find the contribution of the quadratic factor of Eq. (8.24) to the transient error, and also use parameters similar to those previously employed, coefficients having the same structure as those of Eq. (8.23a) will be substituted for the coefficients n and k of this quadratic factor:

$$(p + m)(p^2 + 2g\omega_g p + \omega_g^2) = 0. \tag{8.27}$$

The two roots of the quadratic factor therefore generally represent a damped oscillatory term of the error, for which the resonant frequency ω_g and damping ratio g correspond, respectively, to ω_n and c of the previous systems,

$$\theta = \epsilon^{-g\omega_g t}(B_1 \cos \omega_g \sqrt{1 - g^2}t + B_2 \sin \omega_g \sqrt{1 - g^2}t) \tag{8.28}$$

or

$$\theta = \epsilon^{-at}(B_1 \cos bt + B_2 \sin bt) \tag{8.29}$$

where

$$a = g\omega_g \tag{8.30}$$

$$b = \omega_g \sqrt{1 - g^2} \tag{8.31}$$

Complete General Solution.—The complete general solution of any differential equation that has Eq. (8.27) as its auxiliary equation is the sum of the two solutions set forth in Eqs. (8.26) and (8.28) or Eqs. (8.26) and (8.29).

$$\theta = A\epsilon^{-mt} + \epsilon^{-g\omega_g t}(B_1 \cos \omega_g \sqrt{1 - g^2}t + B_2 \sin \omega_g \sqrt{1 - g^2}t) \tag{8.32}$$

or

$$\theta = A\epsilon^{-mt} + \epsilon^{-at}(B_1 \cos bt + B_2 \sin bt). \tag{8.33}$$

At this point it can be seen that it is only necessary to find the relations between ω_g, g, and m, and ω_n, c, and s to be able to use Eq. (8.32) or Eq. (8.33) as a solution of the original third-order equation (8.17). In order to correlate these six constants, Eq. (8.27) is expanded as follows:

$$p^3 + (2g\omega_g + m)p^2 + (2gm + \omega_g)\omega_g p + m\omega_g^2 = 0. \tag{8.34}$$

For this equation to be identical to Eq. (8.23) it is necessary that the corresponding coefficients of p be equal. Thus, equating the coefficients of these equations,

$$2g\omega_g + m = 2c\omega_n \tag{8.35}$$
$$(2gm + \omega_g)\omega_g = \omega_n^2 \tag{8.36}$$
$$m\omega_g^2 = s\omega_n^3 \tag{8.37}$$

Comparison of the two members of each of these equations shows that m has the dimensions of a radian frequency. Employing the practice previously followed of using dimensionless parameters, m may therefore be replaced by the product of the frequency[1] ω_g and a dimensionless constant h defined by the equation

$$m = h\omega_g. \tag{8.38}$$

Eqs. (8.35), (8.36), and (8.37) then become

$$(2g + h)\omega_g = 2c\omega_n \tag{8.39}$$
$$(2gh + 1)\omega_g^2 = \omega_n^2 \tag{8.40}$$
$$h\omega_g^3 = s\omega_n^3 \tag{8.41}$$

The parameter h may be considered as a modified integral proportionality constant.

Combining Eqs. (8.38) and (8.32), the transient solution is expressed

$$\theta = A\epsilon^{-h\omega_g t} + \epsilon^{-g\omega_g t}(B_1 \cos \omega_g \sqrt{1 - g^2}t + B_2 \sin \omega_g \sqrt{1 - g^2}t) \tag{8.42}$$

Applying Eqs. (8.11), (8.12), (8.13), and (8.15), the second members of Eqs. (8.39), (8.40), and (8.41) may be replaced by their expressions in terms of the system constants F, L, J, K, and N. Equations (8.39), (8.40), (8.41) then become

$$(2g + h)\omega_g = \frac{L + F}{J} \tag{8.43}$$

$$(2gh + 1)\omega_g^2 = \frac{K}{J} \tag{8.44}$$

$$h\omega_g^3 = \frac{N}{J}. \tag{8.45}$$

[1] The frequency ω_g is chosen rather than ω_n or any other frequency in order that the equations may appear in the most convenient form.

Furthermore, solution of Eqs. (8.39), (8.40), and (8.41) for c, s, and ω_g/ω_n gives

$$c = \frac{2g + h}{2(2gh + 1)^{1/2}} \tag{8.46}$$

$$s = \frac{h}{(2gh + 1)^{3/2}} \tag{8.47}$$

$$\frac{\omega_g}{\omega_n} = \frac{1}{(2gh + 1)^{1/2}}. \tag{8.48}$$

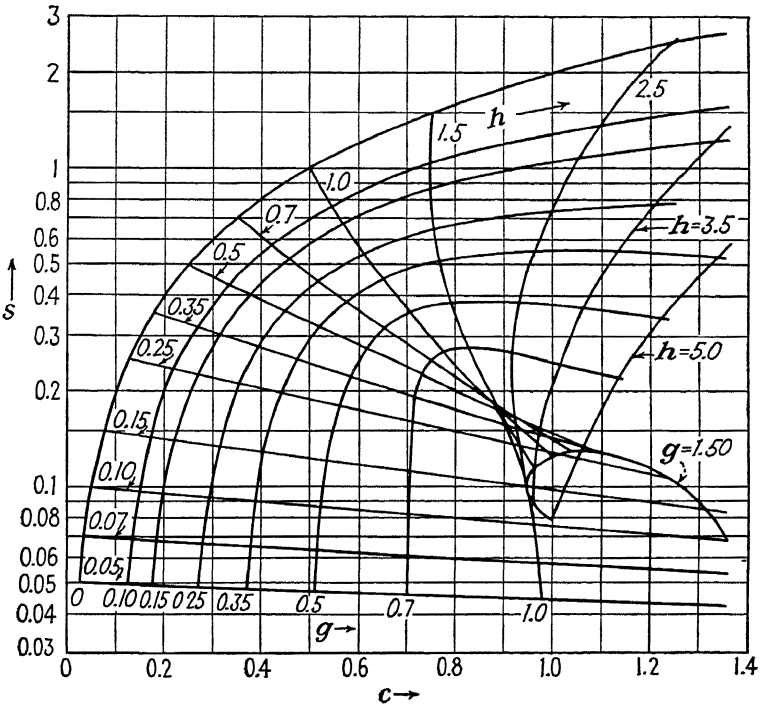

FIG. 8.4.—Effect of integral control on oscillatory component damping constant and exponential component decay rate.

These relationships are represented graphically by the curves of Figs. 8.4 and 8.5. From the first set of curves (Fig. 8.4) values of the modified damping ratio g and integral control factor h can be obtained for any given values of the original damping ratio c and integral control factor s of the system. From the values of g and h thus found, it is possible to find the corresponding value of the ratio ω_g/ω_n by referring to Fig. 8.5. Knowing the value of ω_n, the error θ can then be calculated, as a function of time, from Eq. (8.33) or Eq. (8.42), after determination of the constants A, B_1, and B_2. This determination will be done presently.

Determination of Constant Coefficients from Boundary Conditions.—As for any third-order differential equation, there are three arbitrary

constants in the general solution, Eq. (8.33), of Eq. (8.17). In order to evaluate these three constants, three boundary conditions are necessary.

1. Up to the starting instant of the input member of the system, the output member displacement and input-output error are zero.

$$\theta = \theta_o = 0, \qquad t = 0. \tag{8.49}$$

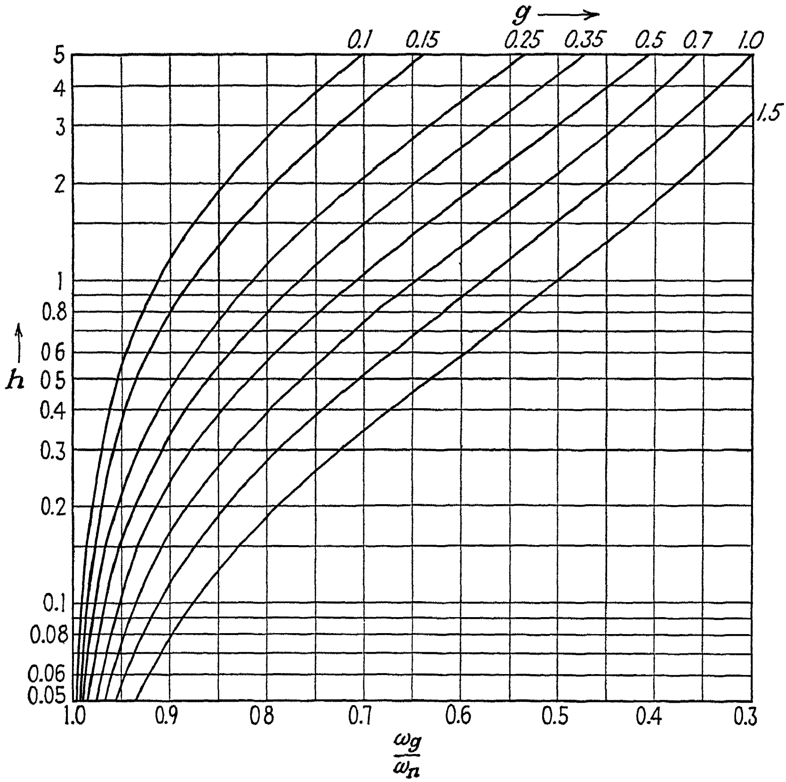

FIG. 8.5.—Effect of integral control on natural frequency.

Applying this condition to Eq. (8.33) for $t = 0$ gives

$$\theta = A + B_1 \tag{8.50}$$

$$A = -B_1. \tag{8.51}$$

2. Since only finite forces may be applied to the output member, the latter cannot be accelerated instantly from rest to any finite velocity. The output displacement and velocity are therefore zero at the instant of time $t = 0$ when the input member is set in motion. Consequently, the error velocity, or first time derivative of the error, is equal to the input velocity, as expressed in Eq. (8.6) at the instant $t = 0$.

$$\frac{d\theta}{dt} = \frac{d\theta_i}{dt} = \omega_1, \qquad t = 0. \tag{8.52}$$

In order to apply this condition to Eq. (8.33), it is first necessary to differentiate this equation.

$$\frac{d\theta}{dt} = -mA\epsilon^{-mt} + \epsilon^{-at}(-B_1b \sin bt + B_2b \cos bt - B_1a \cos bt - B_2a \sin bt) \quad (8.53)$$

which, for $t = 0$, reduces to

$$\frac{d\theta}{dt} = -Am + B_2b - B_1a. \quad (8.53a)$$

Applying then the condition, Eq. (8.52), Eq. (8.53*a*) becomes

$$\omega_1 = -Am + B_2b - B_1a, \quad (8.54)$$

or

$$B_2 = \frac{1}{b}(\omega_1 + Am + B_1a). \quad (8.55)$$

3. A third condition is that the torque developed by the controller, after the transient has died out, must equal the output retarding torque. It was previously noted that no error can exist in the system under steady-state operating conditions. Therefore, the first three terms of Eq. (8.8) become equal to zero, and any steady-state controller torque must be represented by the integral term of this equation. In a servomechanism with inertia loading and viscous output damping, as considered here, output retarding torque results from the viscous output friction, represented by the right-hand term of Eq. (8.8). Therefore, in the steady-state operating condition,

$$F\omega_1 = N \int_0^\infty \theta \, dt. \quad (8.56)$$

Introducing the ratio r of viscous output friction to total damping, as in previous chapters,

$$r = \frac{F}{L + F}, \quad (8.57)$$

and substituting this value in Eq. (8.56) gives

$$(L + F)r\omega_1 = N \int_0^\infty \theta \, dt. \quad (8.58)$$

By substituting Eqs. (8.43) and (8.45) in Eq. (8.58), this becomes

$$(2g + h)r\omega_1 J\omega_g = h\omega_g^3 J \int_0^\infty \theta \, dt, \quad (8.59)$$

or, after dividing through by $J\omega_g$

$$(2g + h)r\omega_1 = h\omega_g^2 \int_0^\infty \theta \, dt. \quad (8.60)$$

The integral term of this equation is evaluated from the form of the error, as expressed in Eq. (8.33).

$$\int_0^\infty \theta\, dt = \int_0^\infty [A\epsilon^{-mt} + \epsilon^{-at}(B_1 \cos bt + B_2 \sin bt)]\, dt. \tag{8.61}$$

Subdividing this integral, the equation becomes

$$\int_0^\infty \theta\, dt = A \int_0^\infty \epsilon^{-mt}\, dt + B_1 \int_0^\infty \epsilon^{-at} \cos bt\, dt + B_2 \int_0^\infty \epsilon^{-at} \sin bt\, dt. \tag{8.62}$$

In this equation, the definite integrals have the following values:

$$\int_0^\infty \epsilon^{-mt}\, dt = \frac{1}{m}; \tag{8.63}$$

$$\int_0^\infty \epsilon^{-at} \cos bt\, dt = \frac{a}{a^2 + b^2}; \tag{8.64}$$

$$\int_0^\infty \epsilon^{-at} \sin bt\, dt = \frac{b}{a^2 + b^2}; \tag{8.65}$$

and Eq. (8.62) can therefore be written

$$\int_0^\infty \theta\, dt = \frac{A}{m} + \frac{B_1 a}{a^2 + b^2} + \frac{B_2 b}{a^2 + b^2} \tag{8.66}$$

or

$$\int_0^\infty \theta\, dt = \frac{A(a^2 + b^2) + B_1 am + B_2 bm}{m(a^2 + b^2)}. \tag{8.67}$$

Substituting for A and B_2 the values found in Eqs. (8.51) and (8.55), this last equation becomes

$$\int_0^\infty \theta\, dt = \frac{-B_1(a^2 + b^2) + 2B_1 am + \omega_1 m - B_1 m^2}{m(a^2 + b^2)}, \tag{8.68}$$

or

$$\int_0^\infty \theta\, dt = \frac{\omega_1 m + B_1(2am - m^2 - a^2 - b^2)}{m(a^2 + b^2)}. \tag{8.69}$$

This may be rewritten in terms of the parameters given in Eqs. (8.30), (8.31), and (8.38) as follows

$$\int_0^\infty \theta\, dt = \frac{h\omega_g\omega_1 + B_1\omega_g^2(2gh - h^2 - 1)}{h\omega_g^3} \tag{8.70}$$

Placing Eq. (8.70) in Eq. (8.60) gives

$$(2g + h)r\omega_1 = h\omega_1 + B_1\omega_g(2gh - h^2 - 1). \tag{8.71}$$

Solving for B_1,

$$B_1 = -\frac{\omega_1[(2g + h)r - h]}{\omega_g(h^2 - 2gh + 1)}. \tag{8.72}$$

Solution for A is obtained by substituting Eq. (8.72) in Eq. (8.51).

$$A = \frac{\omega_1[(2g + h)r - h]}{\omega_g(h^2 - 2gh + 1)}. \tag{8.73}$$

Solution for B_2 is obtained by substituting Eq. (8.72) and (8.73) in Eq. (8.55).

$$B_2 = \frac{1}{b}\left\{\omega_1 + \frac{m\omega_1[(2g + h)r - h]}{\omega_g(h^2 - 2gh + 1)} - \frac{a\omega_1[(2g + h)r - h]}{\omega_g(h^2 - 2gh + 1)}\right\}. \tag{8.74}$$

Substituting for a, b, and m the values given in Eq. (8.30), (8.31), and (8.38), there results

$$B_2 = \frac{\omega_1}{\omega_g\sqrt{1 - g^2}}\left\{1 + \frac{h[(2g + h)r - h]}{h^2 - 2gh + 1} - \frac{g[(2g + h)r - h]}{h^2 - 2gh + 1}\right\} \tag{8.75}$$

$$= \frac{\omega_1}{\omega_g\sqrt{1 - g^2}}\left\{1 + \frac{(h - g)[(2g + h)r - h]}{h^2 - 2gh + 1}\right\} \tag{8.76}$$

$$= \frac{\omega_1}{\omega_g\sqrt{1 - g^2}}\left[\frac{1 - gh + (h - g)(2g + h)r}{h^2 - 2gh + 1}\right]. \tag{8.77}$$

Dimensionless Form of Equations.—Summarizing the preceding developments, when a servomechanism represented by the equation

$$J\frac{d^2\theta}{dt^2} + (L + F)\frac{d\theta}{dt} + K\theta + N\int \theta\, dt = J\frac{d^2\theta_i}{dt^2} + F\frac{d\theta_i}{dt} \tag{8.3}$$

is subjected to a step input velocity function of the form

$$\theta_i = \omega_1 t, \qquad t \geqq 0, \tag{8.5}$$

the input-output error produced, as written from Eq. (8.33) in dimensionless form, is

$$\theta\frac{\omega_g}{\omega_1} = A'\epsilon^{-mt} + \epsilon^{-at}(B'_1 \cos bt + B'_2 \sin bt) \tag{8.78}$$

where, from Eqs. (8.72), (8.73), and (8.77),

$$\left\{\begin{aligned} A' &= \frac{(2g + h)r - h}{h^2 - 2gh + 1}, && (8.79)\\ B'_1 &= -\frac{(2g + h)r - h}{h^2 - 2gh + 1}, && (8.80)\\ B'_2 &= \frac{1}{\sqrt{1 - g^2}}\left[\frac{1 - gh + (h - g)(2g + h)r}{h^2 - 2gh + 1}\right]; && (8.81)\end{aligned}\right.$$

and

$$\left\{\begin{aligned} a &= g\omega_g, && (8.30)\\ b &= \omega_g\sqrt{1 - g^2}, && (8.31)\\ m &= h\omega_g; && (8.38)\end{aligned}\right.$$

and

$$\begin{cases} (2g + h)\omega_g = 2c\omega_n = \dfrac{L + F}{J} & (8.39),(8.43) \\ (2g + h)\omega_g^2 = \omega_n^2 = \dfrac{K}{J} & (8.40),(8.44) \\ h\omega_g^3 = s\omega_n^3 = \dfrac{N}{J}. & (8.41),(8.45) \end{cases}$$

The last three relations may be rewritten as

$$c = \frac{2g + h}{2(2gh + 1)^{1/2}}, \tag{8.46}$$

$$s = \frac{h}{(2gh + 1)^{3/2}}, \tag{8.47}$$

$$\frac{\omega_g}{\omega_n} = \frac{1}{(2gh + 1)^{1/2}}. \tag{8.48}$$

Also

$$r = \frac{F}{L + F}. \tag{8.57}$$

Graphical Interpretation.—Equation (8.78) represents, as a function of time and in dimensionless form, the input-output position error of a

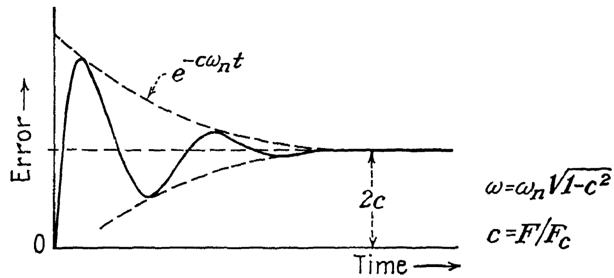

FIG. 8.6.—Error-time curve for viscous-damped servomechanism subjected to step input velocity function.

damped servomechanism with integral control, subjected to a step input function.[1] This equation should be compared to corresponding equations (4.65), (5.34), and (6.37), which represent the error in a servomechanism with, respectively, viscous damping, error-rate damping, and combined viscous and error-rate damping. These three equations were represented graphically by the curves of Figs. 4.6, 5.8, and 6.4, which are reproduced here for convenience as Figs. 8.6, 8.7 and 8.8, in abbreviated form.

[1] As in the previous chapters, practical application of this equation requires that the units in which θ, ω_g, ω_1, and t are expressed shall be chosen so as to make the equation dimensionless. This was explained in detail in Chap. IV.

In these three cases it was found that the error comprises (1) an oscillatory component that decays exponentially with time, as expressed by the factor $\epsilon^{-c\omega_n t}$ of the first term of the respective equation, and (2) a steady-state component equal to $2rc$, where $r = 1$ for the purely viscous-damped system and $r = 0$ for the purely error-rate-damped system.

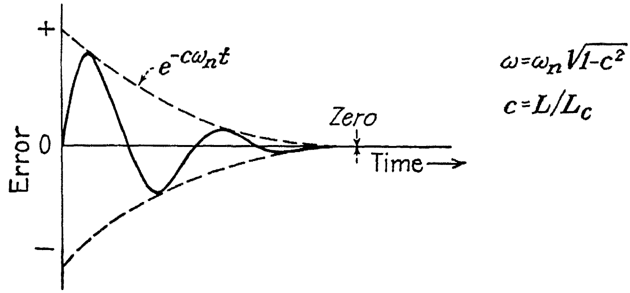

FIG. 8.7.—Error-time curve for error-rate-damped servomechanism subjected to step input velocity function.

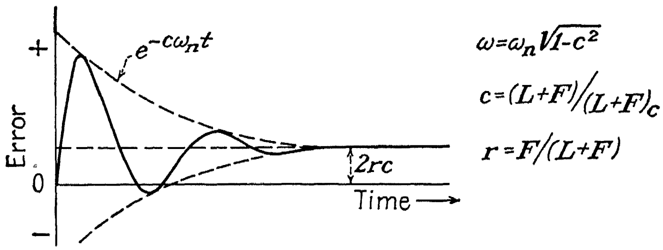

FIG. 8.8.—Error-time curve for servomechanism with combined viscous and error-rate damping subjected to step input velocity function.

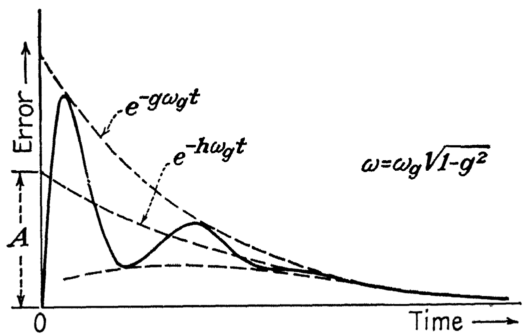

FIG. 8.9.—Error-time curve for damped servomechanism with integral control.

In the case of a viscous-damped system (with or without added error-rate damping) and with integral control, as expressed by Eq. (8.78), the error is represented by the curve of Fig. 8.9. The first term of the equation is nonoscillatory and decays exponentially, as expressed by the factor ϵ^{-mt}. The second term is oscillatory and decays exponentially

at a rate expressed by the factor ϵ^{-at}. Both terms tend toward zero with increasing values of time. For practical reasons, it is generally desirable that both terms become vanishingly small about the same time, or in other words, that the two exponential factors m and a be about equal. Thus, if m is substantially smaller than a, the oscillation will have died out well before the nonoscillatory component, leaving a still sizable error. Conversely, if m is appreciably greater than a, the error will

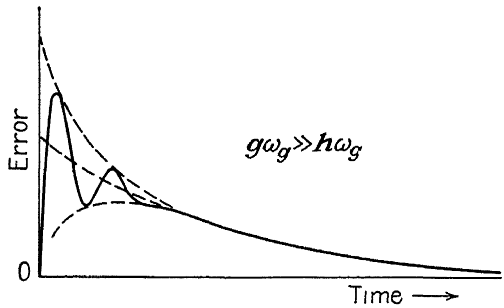

FIG. 8.10.—Error-time curve for damped servomechanism with integral control.

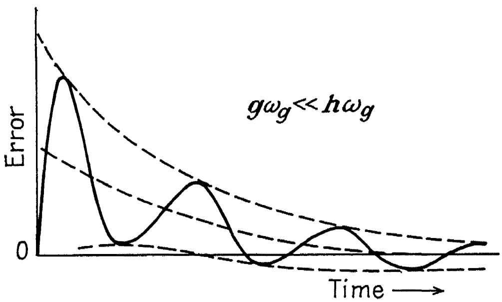

FIG. 8.11.—Error-time curve for damped servomechanism with integral control.

continue oscillating about the zero value after the nonoscillatory component has reduced to zero.

Values frequently used in practice are

$$g = 0.25$$
$$h = 0.15$$

for which the curves of Fig. 8.4 show that

$$c = 0.32$$
$$s = 0.13.$$

For a servo system with pure viscous output damping ($r = 1$), the first (nonoscillatory) and second (oscillatory) terms of Eq. (8.78) are

then equal, respectively, to

$$A' = 0.53$$
$$B' = \sqrt{B_1'^2 + B_2'^2} = 1.12.$$

Since the nonoscillatory term is of substantially smaller magnitude than the oscillatory term, it is permissible to let it decay more slowly than the latter, as actually occurs with the above-mentioned values of h and g.

Another set of values giving satisfactory performance is

$$g = 0.35$$
$$h = 0.25$$

for which the curves of Fig. 8.4 give

$$c = 0.44$$
$$s = 0.2.$$

The amplitudes of the two error components are then

$$A' = 0.8$$
$$B' = 1.27.$$

It is recalled that integral control is intended *not* to stabilize the servomechanism, but to reduce the steady-state error, even when an energy-consuming external load is placed on the output member. Indeed, as may be seen from the above equation (8.78) and the curves (Fig. 8.4) that correlate the constants of that equation, integral control introduces a degree of instability into the system. This may be explained by the regenerative action of the integral control process. Addition of integral control to a system may require that additional stabilization, such as error-rate damping, be incorporated into the system. These points will be developed further in the following chapter, when the transfer functions of integral control servomechanisms are discussed.

Problem.—In order to illustrate some of the relationships between the various parameters involved, consider first a purely viscous output damped servo system with the following characteristics:

$$c = 0.25 \qquad K = 100$$
$$\omega_n = 10 \qquad J = 1$$
$$F = 2c\sqrt{KJ} = 5.$$

As recalled in the preceding paragraph, the dimensionless steady-state error of this servo will be equal to $2c = 0.5$, and the rate of decay of the oscillation is characterized by the product

$$c\omega_n = 2.5.$$

Let integral control now be added to the system, and identify this by an integral control constant $s = 0.25$. The curves of Fig. 8.4 show that for

$$c = 0.25 \qquad \text{and} \qquad s = 0.25$$

the following values obtain

$$g = 0.12 \qquad h = 0.27.$$

This means that the effective damping of the oscillatory component is reduced to 0.12, which can be recognized as a rather unstable system. Referring then to Fig. 8.5, these values of g and h are found to correspond to

$$\frac{\omega_g}{\omega_n} = 0.97.$$

Thus, the new oscillation frequency is

$$\omega_g = 0.97 \times 10 = 9.7$$

or almost the same as it was without integral control. However, the rate of decay of the oscillation when integral control is used is now characterized by the product

$$g\omega_g = 0.12 \times 9.7 = 1.16,$$

which is substantially less than in the original system: the oscillation is damped out more slowly.

In order to increase the damping rate of the oscillation, it is necessary to increase the value of g. Since the friction F and inertia J of the system are fixed, this can be done by changing the controller constant K.

Let then the following values, related to the integral control, be arbitrarily set

$$g = 0.25 \qquad s = 0.25 \qquad h = 0.27,$$

the value of g being now made equal to the value of c as given above. The curves of Fig. 8.4 show that these values are equivalent to a system without integral control, having a damping ratio

$$c = 0.36$$

and a natural frequency

$$\omega_n = \frac{F}{2cJ} = \frac{5}{2 \times 0.36 \times 1} = 6.9.$$

From the relation

$$\omega_n = \sqrt{\frac{K}{J}}$$

it is then found that K must be equal to 47.6 instead of 100 as originally given. From the curves of Fig 8.5, it is seen that

$$\frac{\omega_g}{\omega_n} = 0.94, \qquad \text{for} \qquad g = 0.25 \qquad \text{and} \qquad h = 0.27.$$

It follows that

$$\omega_g = 6.9 \times 0.94 = 6.48.$$

Thus, while the oscillation frequency is now 6.48 instead of 10 as in the original system, its rate of decay has been increased to 1.6. Naturally, if there is true integral control, no steady-state error will exist.

Dependence of Torque on Error Frequency in a Viscous-damped Servo with Integral Control.—While the steady-state error in a servomechanism with integral control is zero when the input member is driven at constant speed and after sufficient time has elapsed for the transient

to die out, such is not the case when the input speed is varied. Thus, if the input member is displaced sinusoidally with time at some frequency, the output member displacement as well as the input-output error are also sinusoidal functions of time of the same frequency.

Consider first, for simplicity, a purely viscous output damped system with integral control but without error-rate damping ($L = 0$). Equation (8.1) shows that the accelerating or driving torque developed by the controller is then

$$T = K\theta + N\int\theta\,dt. \tag{8.82}$$

Let then the input member be displaced sinusoidally, so that the error is a sinusoidal function of time of unit amplitude

$$\theta = \sin \omega t. \tag{8.83}$$

The integral of the error is

$$\int \theta\,dt = -\frac{1}{\omega}\cos \omega t, \tag{8.84}$$

and the controller torque is expressed

$$\begin{aligned} T &= K \sin \omega t - \frac{N}{\omega}\cos \omega t \\ &= K\left(\sin \omega t - \frac{N}{K\omega}\cos \omega t\right). \end{aligned} \tag{8.85}$$

Combining Eqs. (8.11) and (8.15), it is found that

$$\frac{N}{K} = s\omega_n \tag{8.86}$$

which substituted in Eq. (8.85) gives for the torque

$$T = K\left(\sin \omega t - \frac{s\omega_n}{\omega}\cos \omega t\right), \tag{8.87}$$

or

$$T = K\left(\sqrt{1 + \frac{s^2\omega_n^{\,2}}{\omega^2}}\sin \omega t + \lambda\right), \tag{8.88}$$

where the phase angle of lag λ of the torque behind the error is defined as

$$\lambda = -\tan^{-1}\frac{s\omega_n}{\omega}. \tag{8.89}$$

This torque as shown in Eq. (8.82) is a function of the error and error time integral. Its magnitude, according to Eq. (8.88), is equal to

$$|T| = K\sqrt{1 + \frac{s^2\omega_n^{\,2}}{\omega^2}} \tag{8.90}$$

and varies with the error frequency ω in the manner shown by the curve of Fig. 8.12.

As the frequency ω tends toward zero, the torque increases and tends toward infinity. As the frequency ω increases, the torque decreases rapidly at first, becomes equal to $K\sqrt{2}$ for a value ω_q of the frequency equal to $\omega_q = s\omega_n$, and then tends asymptotically toward the value K as the frequency becomes increasingly great.

Thus, the effect of integral control is particularly noticeable at the lower frequencies, where the integral term of Eq. (8.82) has considerable magnitude, as expressed in Eq. (8.84). At the higher frequencies this term becomes negligibly small with respect to the error proportional term $K\theta$, which remains constant independent of the frequency.

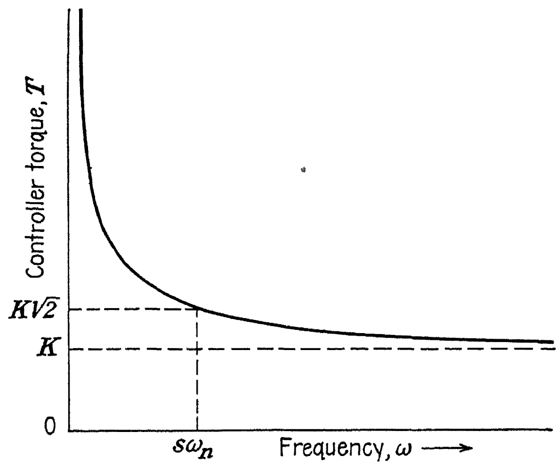

Fig. 8.12.—Theoretical torque-frequency characteristic of integral control.

Error Integrating Control Circuits.—In the preceding discussion the controller torque was expressed in terms of the error magnitude and frequency. If a mechanical differential device is used, a suitable corresponding electrical error signal will be obtained by letting the differential error shaft of the device drive a potentiometer connected across a d-c source. If a synchro repeater follow-up link is used between the input and output members of the system, a similar voltage is obtained by rectifying the alternating (line frequency) error voltage before applying it to the controller. In both cases the error voltage then has the same frequency as the input-output position error and a magnitude proportional to that of the error.

In addition to the error voltage just described, a signal proportional to the time integral of the error voltage must be conveyed to the input terminals of the controller amplifier. An integrating circuit suitable for

this purpose is shown schematically in Fig. 8.13. It consists of a resistor R_1 in series with the error signal line and a series combination of a resistor R_2 and capacitor C_2 connected across the error signal line. Other parts of the system comprise a d-c controller amplifier, a d-c servo motor, a friction damper and output inertia load, synchro repeaters, and an a-c rectifier placed between the synchro differential and the error signal integrating network $R_1R_2C_2$.

For a constant or very low frequency error the voltage output of this integrating network, as applied to the input terminals of the amplifier,

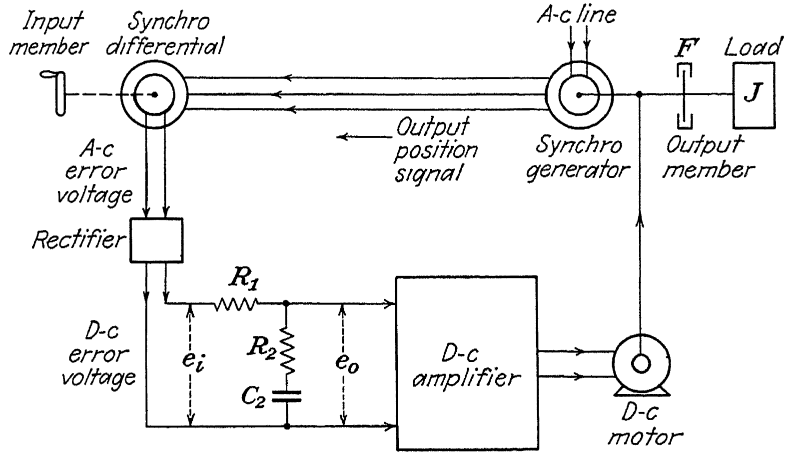

FIG. 8.13.—Servomechanism with error-integrating network.

is equal to the error voltage applied to the input terminals of the network. For higher frequency error voltages the capacitor C_2 has almost negligibly small impedance, and the voltage applied to the amplifier input terminals is a fraction [roughly equal to $R_2/(R_1 + R_2)$] of the error voltage applied to the input terminals of the integrating network.

A more accurate formulation of the operation of the network is obtained from a simple analysis of the circuit. Referring to Fig. 8.13, let the input voltage e_i of the network be an alternating voltage having a peak value of 1 volt. The output voltage e_o of the network is then

$$e_o = \frac{R_2 + \dfrac{1}{j\omega C_2}}{R_1 + R_2 + \dfrac{1}{j\omega C_2}} = \frac{j\omega R_2 C_2 + 1}{j\omega R_1 C_2 + j\omega R_2 C_2 + 1} \tag{8.91}$$

For simpler writing, let

$$M = \frac{R_1}{R_2} \tag{8.92}$$

and

$$\omega_q = \frac{1}{R_2 C_2}. \tag{8.93}$$

Equation (8.91) then becomes

$$e_o = \frac{j\dfrac{\omega}{\omega_q} + 1}{j\dfrac{\omega M}{\omega_q} + j\dfrac{\omega}{\omega_q} + 1}. \tag{8.94}$$

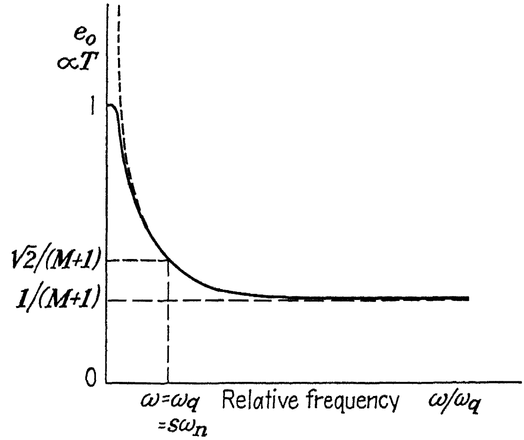

FIG. 8.14.—Transfer characteristic of integrating network.

The magnitude of this complex output voltage of the network is expressed

$$|e_o| = \sqrt{\frac{1 + (\omega^2/\omega_q^2)}{1 + (M+1)^2\dfrac{\omega^2}{\omega_q^2}}}. \tag{8.95}$$

This expression is represented graphically as a function of ω/ω_q by the solid line curve of Fig. 8.14. For comparison the theoretical curve of Fig. 8.12 is reproduced in dotted line.[1] It is seen that, when the frequency ω

[1] The previous theoretical discussion showed that the controller torque should become infinitely large at zero frequency. As illustrated in Fig. 8.14, the torque indeed becomes very large, but remains of finite value. This is because of unavoidable physical limitations of the actual components of the system, with the result that in practice the steady-state error is not completely corrected to zero.

In the pendulum analogy represented in Fig. 8.2, a corresponding condition may be simulated by assuming that a small amount of sand is constantly leaking out of the cup D. As the weight of the sand in the cup decreases, the pendulum is no longer maintained in exactly vertical position against the viscous drag produced by the translatory motion of the pendulum vane in the damping fluid. As the pendulum then assumes a backward slanting position (Fig. 8.2a), the funnel H opens and allows sand to flow from the carriage into the cup D. The position of equilibrium is that at which the flow of sand into the cup is equal to the leakage flow out of the cup. This in turn implies the existence of a steady-state error, small as this may be. In other words, practical limitations of the system result in corresponding limitations in the process of integral control.

is small, the expression for the output voltage of the network, as given in Eq. (8.95), reduces to

$$|e_o| = 1 \text{ volt.} \tag{8.96}$$

When the frequency ω is large, this voltage becomes

$$|e_o| = \frac{1}{M+1} \tag{8.97}$$

If the resistance ratio of the network is large (*e.g.*, of the order of $M = 10$), a frequency $\omega = \omega_q$ will give an output voltage

$$|e_o| = \sqrt{\frac{1+1}{1+(M+1)^2}} \cong \frac{\sqrt{2}}{M+1} \tag{8.98}$$

According to the remarks following Eq. (8.90), this frequency ω_q is therefore equal to

$$\omega_q = s\omega_n. \tag{8.99}$$

Problem.—A servo system with viscous output damping has the following characteristics:

$F = 50 \times 10^{-6}$ ft.-lb. per radian per sec. at the motor shaft.
$J = 4 \times 10^{-6}$ slug-ft.^{2} at the motor shaft.
$\omega_n = 4$ cycles per sec. $= 25$ radians per sec., approximately.
$K = 25$ ft.-lb. per radian error at the output shaft.
$\theta_s =$ steady-state error at 10 r.p.m. $= 1.2$ degs.

Integral control is then introduced in the system by inserting a suitable network in the error signal channel in the manner shown in Fig. 8.13. Choosing

$$\frac{R_1}{R_2} = M = 9 \qquad \text{and} \qquad C_2 = 1 \text{ mf.}$$

calculate the values of R_1 and R_2 and the steady-state error of the system. A gear of 100:1 ratio is placed between the motor and output shaft.

Solution: Referring F and J to the output shaft, these become

$F_o = 50 \times 10^{-6} \times 100^2 = 0.5$ ft.-lb. per radian per sec.
$J_o = 4 \times 10^{-6} \times 100^2 \doteq 0.04$ slug-ft.^{2}

The damping ratio of the system, before application of integral control, is then

$$c = \frac{F}{2\sqrt{KJ}} = \frac{0.5}{2\sqrt{25 \times 0.04}} = 0.25.$$

The curves of Fig. 8.4 show that for this value of c and an integral control factor arbitrarily chosen as

$$s = 0.2$$

the factors g and h have barely acceptable values of approximately comparable magnitudes. Equation (8.99) then gives

$$\omega_q = s\omega_n = 0.2 \times 25 = 5 \text{ radians per sec.}$$

At this frequency the gain of the network must be $\sqrt{2}$ times as great as at substantially higher frequencies.

Now, from Eq. (8.93),

$$\omega_q = 5 = \frac{1}{R_2C_2} = \frac{1}{R_2 \times 1 \times 10^{-6}}$$

from which

$$R_2 = \frac{1 \times 10^{-6}}{5} = 200{,}000 \text{ ohms}$$

and

$$R_1 = 9R_2 = 9 \times 200{,}000 = 1.8 \text{ megohms.}$$

Since the ratio $M = R_1/R_2$ is equal to 9, the presence of the integrating circuit in the error signal channel introduces, according to Eq. (8.97), an error voltage loss of 10 to 1. That is to say, the voltage applied to the amplifier input terminals is one-tenth the total error voltage applied to the input terminals of the integrating network.

In order to compensate for this loss, it is necessary to increase the amplifier gain by a factor of 10. On the other hand, the retarding torque $F\omega_1$ is, in the steady-state, equal to the accelerating torque $K\theta$. This retarding torque being constant, for a given operating speed ω_1 and friction F, the ten-fold increase of the effectiveness of the controller constant K implies a ten-fold decrease of the steady-state error θ. Instead of being equal to 1.2 deg., the steady-state error is therefore reduced to 0.12 deg.

Problem.—Consider a viscous output damped servomechanism intended to be used for orienting or continuously rotating a mechanical load, such as a radio antenna, for instance. Use is made of a servo motor having a moment of inertia of 100×10^{-6} slug-ft.^{2} and a locked torque of 1.0 ft.-lb. The torque-speed characteristic of the motor shows a decrease in torque of 30 per cent at 2,000 r.p.m. A 400:1 step-down gear ratio is provided between the motor and load, so that this may be driven at a speed of 5 r.p.m. The damping ratio of the system is $c = 0.25$.

Disregarding the inertia of the load, the moment of inertia at the output shaft is

$$J = 100 \times 10^{-6} \times 400^2 = 16 \text{ slug-ft.}^2$$

The friction torque provided by the motor characteristic is

$$\frac{\text{Torque variation}}{\text{Speed variation}} = \frac{0.3 \text{ ft.-lb.}}{200 \text{ radians per sec.}} = 0.0015 \text{ ft.-lb. per radian per second}$$

which, at the output shaft, is equal to

$$F = 0.0015 \times 400^2 = 240 \text{ ft.-lb. per radian per second}$$

The natural frequency of the system is then

$$\omega_n = \frac{F}{2cJ} = \frac{240}{2 \times 0.25 \times 16} = 30 \text{ radians per sec.}$$
$$= \text{about 5 cycles per sec.}$$

The steady-state error at a speed of 5 r.p.m. (or 0.5 radian per sec.) is equal to

$$\theta_s = \frac{2c\omega_1}{\omega_n} = \frac{2 \times 0.25 \times 0.5}{30} \times 57.3 = 0.5 \text{ deg.}$$

From the torque-speed characteristic of the motor it is seen that the motor torque at 2,000 r.p.m. is equal to 0.7 ft.-lb., which is equivalent to $0.7 \times 400 = 280$ ft.-lb. at

the load shaft. Suppose that the load is subjected to wind, which causes a torque of 200 ft.-lb. Calculate the angular lag of the load.

The controller factor K is equal to

$$K = \omega_n^2 J = 30^2 \times 16 = 14{,}400 \text{ ft.-lb. per radian.}$$

Applying the relation

$$\text{Output torque} = K\theta = 200 \text{ ft.-lb.}$$

the error due to the wind load is

$$\theta_w = \frac{200}{K} = \frac{200}{14{,}400} \times 57.3 = \frac{5}{6} \text{ deg.}$$

The total error is therefore equal to

$$\theta = \theta_s + \theta_w = \tfrac{1}{2} + \tfrac{5}{6} = 1\tfrac{1}{3} \text{ deg.}$$

It should be noted that at zero speed the ½-deg. error disappears, leaving, however, the wind load error of ⅚ deg.

In order to reduce the error, two methods are available.

1. The value of the controller constant K being kept unchanged at the natural oscillation frequency, the effectiveness of K may be increased at the lower frequencies by introducing integral control into the system. This is done by inserting in the error signal channel a rectifier and integrating network, as shown in Fig. 8.13. If the error is to be reduced to one-fifth its original value, the *effect* of K must be increased by a factor of 5. The resistance ratio $M = R_1/R_2$ of the integrating network is then made equal to 4, and the gain at low frequencies is five times as great as at high frequencies.

Since the damping ratio c is equal to 0.25, a suitable value for the integral control factor may be chosen as $s = 0.15$. For these values of c and s the curves of Fig. 8.4 show a value $g = 0.15$ for the new stability coefficient of the system, which, in effect, takes the place of the factor c of the original system.

The frequency ω_q, as defined before, is then equal to

$$\omega_q = s\omega_n = 0.15 \times 30 = 4.5.$$

If the value of the capacitor C_2 is chosen as 1 mf., the relations

$$\begin{cases} \omega_q = \dfrac{1}{R_2 C_2} = 4.5 \\ M = \dfrac{R_1}{R_2} = 4 \end{cases}$$

allow the resistance values of the two resistors to be determined.

$$R_2 = \frac{1}{\omega_q C_2} = \frac{1}{4.5 \times 1 \times 10^{-6}} = 220{,}000 \text{ ohms,}$$
$$R_1 = 4R_2 = 880{,}000 \text{ ohms.}$$

Since neither K nor J has been changed, the natural frequency ω_n of the system remains substantially the same; but the output torque is increased by a factor of 5, and the error is reduced to one-fifth its original value. However, the system is quite unstable.

2. While the above method reduces the steady-state error of the system, it may be desired also to reduce the transient error in the same proportion. This can be done

by adding error-rate damping. However, this increased damping would require the controller constant K to be multiplied by $\sqrt{5}$. This in turn raises the natural frequency to a value $\omega_n\sqrt{5}$, or about 67 radians per sec. (10 cycles per sec.). This may be objectionably high, since in practice the natural frequency should not exceed some 15 per cent of the carrier (line) frequency of the synchro excitation voltage.

Dependence of Torque on Frequency in a Servo with Combined Error-rate Damping and Integral Control.—The previous discussion considered the use of integral control in a viscous output damped servomechanism. The case of a purely error-rate-damped servo will be taken up now. Absence of viscous damping implies, of course, that the motor torque is independent of the motor speed and depends only on the motor control voltage.

Equation (8.1) shows that the torque developed by the controller is expressed

$$T = K\theta + L\frac{d\theta}{dt} + N\int \theta\,dt, \tag{8.100}$$

which may also be written

$$T = K\left[\theta + \frac{L}{K}\frac{d\theta}{dt} + \frac{N}{K}\int \theta\,dt\right]. \tag{8.101}$$

In order to study the dependence of the torque on the operating frequency, it may be assumed that the error is a sinusoidal function of time

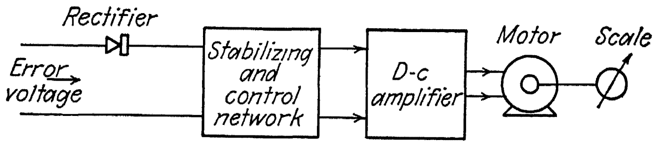

Fig. 8.15.—Test setup for measurement of controller torque-frequency characteristic.

of unit amplitude. In practice this may be accomplished experimentally, as shown in Fig. 8.15, by breaking the servo loop, feeding a suitably modulated voltage into the error channel, and connecting a scale to the motor shaft to measure the torque.

The error being expressed as

$$\theta = \sin \omega t,$$

its first time derivative and its time integral are

$$\frac{d\theta}{dt} = \omega \cos \omega t,$$

$$\int \theta\,dt = -\frac{\cos \omega t}{\omega}.$$

The torque equation then becomes

$$T = K\left[\sin \omega t + \left(\frac{\omega L}{K} - \frac{N}{\omega K}\right)\cos \omega t\right]; \tag{8.102}$$

and since, as found previously,

$$\frac{L}{K} = \frac{1}{\omega_b} \quad \text{and} \quad \frac{N}{K} = \omega_q$$

the equation becomes

$$T = K\left[\sin \omega t + \left(\frac{\omega}{\omega_b} - \frac{\omega_q}{\omega}\right)\cos \omega t\right], \tag{8.103}$$

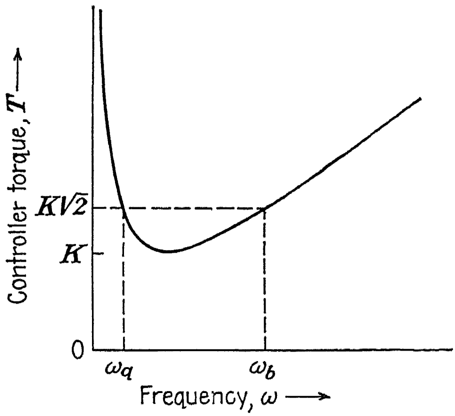

FIG. 8.16.—Theoretical torque-frequency characteristic with error-rate damping and integral control.

or

$$\begin{cases} T = K\left[1 + \left(\dfrac{\omega}{\omega_b} - \dfrac{\omega_q}{\omega}\right)^2\right]^{1/2} \sin(\omega t + \lambda) & (8.104) \\ \lambda = \tan^{-1}\left(\dfrac{\omega}{\omega_b} - \dfrac{\omega_q}{\omega}\right) & (8.105) \end{cases}$$

The magnitude of the controller torque, as expressed in Eq. (8.104), is represented graphically as a function of error frequency in the graph of Fig. 8.16. The torque is a minimum equal to K when

$$\frac{\omega}{\omega_b} = \frac{\omega_q}{\omega} \quad \text{or} \quad \omega = \sqrt{\omega_b \omega_q} \tag{8.106}$$

At the higher frequencies the torque tends to increase directly as the frequency, as in the case of simple error-rate damping. At the lower frequencies the torque increases toward infinity as a result of integral control. If ω_b is substantially greater than ω_q, the torque is equal to $K\sqrt{2}$ when the error frequency ω is equal to either ω_b or ω_q.

A network providing both error-rate damping and integral control is illustrated in Fig. 8.17. Its mode of operation can be understood from a simple analysis of the circuit. First express the two impedances Z_1 and Z_2 of the network,

$$Z_1 = \frac{1}{\frac{1}{R_1} + j\omega C_1} = \frac{R_1}{1 + j\omega R_1 C_1} \tag{8.107}$$

$$Z_2 = R_2 + \frac{1}{j\omega C_2} = \frac{1 + j\omega R_2 C_2}{j\omega C_2}. \tag{8.108}$$

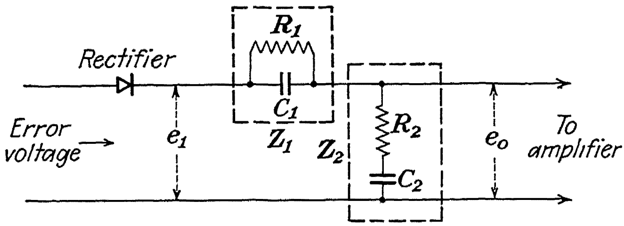

FIG. 8.17.—Differentiating and integrating network for combined error-rate damping and integral control.

For an input voltage of 1 volt the output voltage is then

$$e_o = \frac{Z_2}{Z_1 + Z_2} = \frac{\dfrac{1 + j\omega R_2 C_2}{j\omega C_2}}{\dfrac{1 + j\omega R_2 C_2}{j\omega C_2} + \dfrac{R_1}{1 + j\omega R_1 C_1}}$$

$$= \frac{(1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2)}{(1 + j\omega R_1 C_1)(1 + j\omega R_2 C_2) + j\omega R_1 C_2}. \tag{8.109}$$

Applying the relations previously found,

$$\begin{cases} \omega_q = \dfrac{1}{R_2 C_2} \\ \omega_b = \dfrac{1}{R_1 C_1} \\ M = \dfrac{R_1}{R_2} \end{cases}$$

Eq. (8.109) becomes

$$e_o = \frac{\left(1 + j\dfrac{\omega}{\omega_q}\right)\left(1 + j\dfrac{\omega}{\omega_b}\right)}{\left(1 + j\dfrac{\omega}{\omega_q}\right)\left(1 + j\dfrac{\omega}{\omega_b}\right) + jM\dfrac{\omega}{\omega_q}}. \tag{8.110}$$

In this equation, the integral factor ω_q is generally much smaller than the error-rate factor ω_b. Consequently for $\omega \ll \omega_b$ the equation reduces to

Eq. (8.94). For $\omega \gg \omega_q$ Eq. (8.110) reduces to

$$e_o = \frac{1 + j\,(\omega\ \omega_b)}{(M + 1) + j(\omega/\omega_b)}; \tag{8.111}$$

and for values of ω/ω_b somewhat less than $(M + 1)$, this equation can be written as

$$e_o = \frac{1 + j(\omega/\omega_b)}{M + 1}. \tag{8.112}$$

The variations of e_o (or of T) with the frequency ω are illustrated by the solid line in Fig. 8.18.

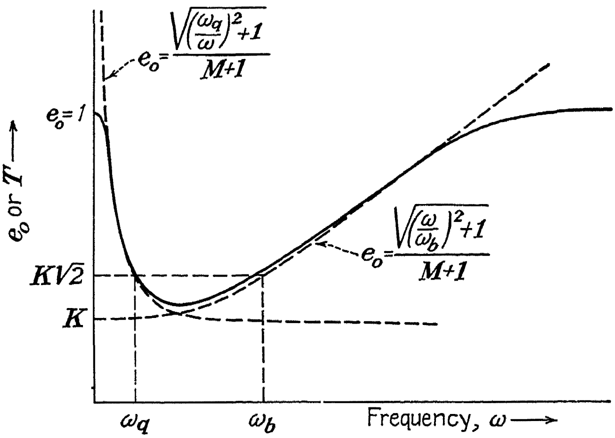

FIG. 8.18.—Transfer characteristic of combined differentiating and integrating network.

Example.—Consider a servomechanism with pure error-rate damping, having a natural frequency $\omega_n = 40$ radians per sec. Integral control is added to the system, by placing in the error voltage channel a network similar to that shown in Fig. 8.17. The desired amount of damping is $g = 0.25$, for which a practically acceptable value of the constant h is 0.15. From the curves of Fig 8.4 these values are seen to correspond to $c = 0.32$ and $s = 0.13$. It follows that

$$\omega_b = \frac{\omega_n}{2c} = \frac{40}{2 \times 0.32} = 60 \text{ radians per se}$$

$$\omega_q = s\omega_n = 0.13 \times 40 = 5 \text{ radians per sec.}$$

If the moment of inertia, measured at the output shaft, is 16 slug-ft.^{2} as in the preceding exercise, the torque amplification constant of the controller is

$$K = \omega_n^2 J = 40^2 \times 16 = 25{,}600 \text{ ft.-lb. per radian error.}$$

Suppose now that the error must be kept below 0.1 deg. with the motor delivering ¼ hp. at 2,000 r.p.m., the motor being geared down 400:1 to the output load. The motor torque is then

$$T_M = \frac{0.25 \times 10^6}{2{,}000} = 125 \text{ in.-oz.}$$

$$T_M = 0.625 \text{ ft.-lb.}$$

Applying the relation

$$\text{Torque} = (M + 1)K\theta$$

and referring the motor torque to the output shaft, the error is expressed

$$\theta = \frac{0.625 \times 400}{25{,}600(M + 1)} = 0.1 \text{ deg.}$$

from which

$$M = \left(\frac{0.625 \times 400}{25{,}600 \times 0.1} \times 57.3\right) - 1 = 5 \text{ (approximately).}$$

Thus, by making $R_1 = 5R_2$, the gain at zero frequency will be six times the gain at the frequency ω_q. Assuming that the capacitor C_2 in Fig. 8.17 is chosen as having a value of 1 mf., the relation

$$\omega_q = \frac{1}{R_2C_2} = 5$$

gives at once

$$R_2 = 200{,}000 \text{ ohms.}$$

From this, the value of R_1 is calculated.

$$R_1 = MR_2 = 5 \times 200{,}000 = 1{,}000{,}000 \text{ ohms.}$$

Finally, from the relation

$$\omega_b = \frac{1}{R_1C_1} = 60$$

the value of C_1 is found to be

$$C_1 = 0.017 \text{ mf.}$$

Integral Control Circuits for Alternating-current Controllers.—The integrating networks discussed in the preceding paragraphs were shown to operate on an error voltage that has the same frequency as the input-output position error itself. In practice, this error frequency is comprised between the limits of zero and some 10 to 15 cycles per sec. The output voltage of the network for a unit input voltage then varies with the frequency in the manner shown by the curve of Fig. 8.14: it is equal to the input voltage at the lower error frequencies, and tends toward the value $1/(M + 1)$ at the higher frequencies. The ratio $1/(M + 1)$ generally has a value of about 0.1. This output voltage is then suitable for actuating a d-c amplifier and servo motor.

Certain difficulties (voltage drift, and the like) inherent in the design of stable d-c amplifiers, and the advantages of using a-c motors of the two-phase induction type, make it often desirable to modify the system and to adapt it for use with a-c amplifiers and motors.

It will be recalled that in a servo system employing a synchro repeater follow-up link the error voltage furnished by the synchros is an alternating voltage of synchro excitation frequency (60 cycles in many applications). This voltage is modulated by the error at its own, substantially lower frequency. Thus, as explained previously, a zero-frequency error

corresponds to an error voltage of 60 cycles frequency (or such other carrier frequency u_c as may be used for exciting the synchros). An error frequency ω different from zero corresponds to an error voltage having *two* frequencies equal, respectively, to $u_c + \omega$ and $u_c - \omega$. In other words, the error frequencies are transposed to the carrier level and extend symmetrically on each side of the carrier frequency.

An integrating circuit operating on such a voltage should therefore have an output voltage characteristic as shown in Fig. 8.19, which is a

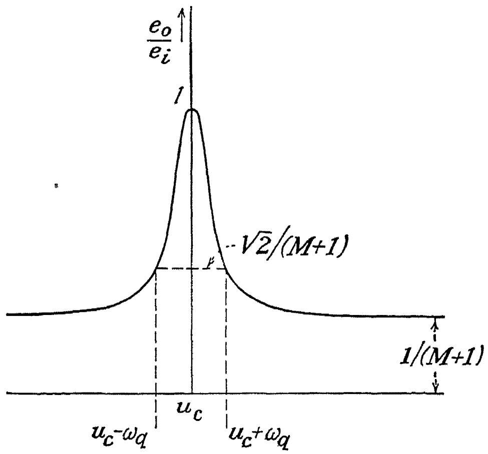

FIG. 8.19.—Transfer characteristic of integrating network for use in modulated a-c carrier error channel.

transposition of the curve of Fig. 8.14, as just explained. A tuned circuit may be used for the purpose, having the sharp frequency selectivity indicated by Eq. (8.99), $\omega_q = s\omega_n$. Thus, for values of ω_n of the order of from 3 to 6 cycles per sec. and of s comprised between 0.1 and 0.3 as are usually encountered in practice, the value of ω_q is of the order of 1 cycle per sec. This is but a small fraction of the carrier frequency (60 cycles or more) generally employed, and makes the system critically sensitive to frequency variations of the synchro excitation supply line.

A preferred practice is, therefore, to demodulate or rectify the error voltage supplied by the synchros before applying it to the integrating network, as shown in Fig. 8.13. The demodulated error voltage, having the same frequency as the error, is then operated on by the integrating network in the manner previously described. The output voltage of the network is then reconverted or transposed to the line-frequency level and finally applied to the a-c amplifier and motor of the controller. The integrating network having already been discussed in the preceding paragraphs, only the two new elements of the system will be described here, *viz.*, the demodulator or rectifier used on the input side of the

integrating network, and the modulator or transposing device placed on its output side.

Rectifier.—The error voltage produced by the synchros may be considered essentially as an alternating voltage of the same frequency as the line, or carrier, voltage that excites the synchros. This error voltage has an amplitude that is proportional to the magnitude of the input-output error of the servomechanism. Moreover, it is either in phase or in phase opposition with the synchro excitation voltage, depending on the sense or direction of the error. In other words, the phase depends on whether the output member displacement is leading or lagging the input member displacement.

A rectifier suitable for the present purpose must, therefore, supply a continuous voltage having a magnitude proportional to that of the error voltage produced by the synchros and a polarity that is indicative of the phase of the synchro voltage (*e.g.*, positive when the output voltage of the synchros is in phase with the synchro excitation, and negative when it is in phase opposition with this).

A typical rectifier circuit fulfilling these requirements is shown schematically in Fig. 8.20. For convenience, the line-voltage frequency

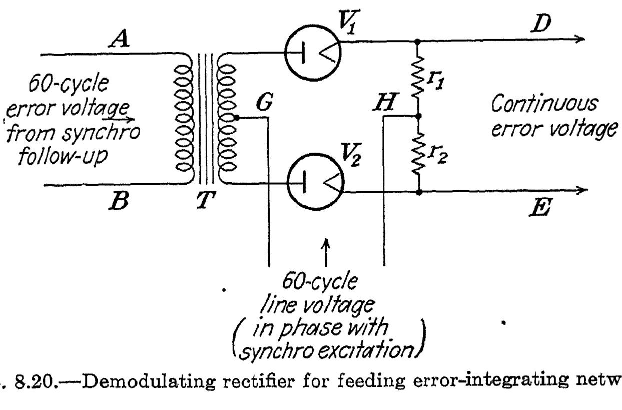

FIG. 8.20.—Demodulating rectifier for feeding error-integrating network.

is designated here as 60 cycles per sec., but it may, of course, have any other suitable value. The 60-cycle error voltage developed by the synchro follow-up applied to the primary terminals A and B of a transformer T. The secondary terminals of this transformer are connected to the plates of two high-vacuum-tube rectifier diodes V_1 and V_2. The filaments of these tubes constitute the output terminals D and E of the device and are connected to each other through resistors r_1 and r_2 in series. The junction point H of these two resistors and a tap G at the center of the transformer secondary winding are connected to the same 60-cycle line that supplies excitation voltage to the synchro repeaters that form the follow-up loop of the servosystem.

Suppose first that there is no error voltage at the points A and B. The only voltage operating on the tubes is then that applied at the terminals G and H. This alternating voltage causes the plates of both tubes simultaneously to become either positive or negative with respect to the filaments. During the negative alternations no current flows through the tubes, and no voltage therefore appears at the output terminals D, E. During the positive alternations both tubes carry equal currents, which in the resistor circuit branch r_1r_2 flow from the filament ends of the resistors toward their common point H. The resulting voltages across these resistors are thus equal and opposite, and here again no voltage appears across the terminals D and E. Thus, in absence of any error voltage across A and B, no voltage is developed across D and E.

If now an error voltage, substantially smaller than the voltage applied across G and H, is applied to the points A and B, the error voltage will cause an alternating voltage to operate on the tubes, in addition to the voltage applied at the terminals G and H and of same frequency as this. This additional error voltage causes the plate of one of the tubes to become more positive than that of the other, and vice versa.

Suppose that, during those alternations of the voltage applied at G and H, which make the plates of both tubes positive, the error voltage at A and B makes the plate of V_1 more positive than that of V_2. Since the current through the tube V_1 is then greater than that through the tube V_2, the output terminal D will be positive with respect to the terminal E. During the other alternations of the voltage applied across G and H, no current flows through the tubes, and no voltage appears at the output terminals D and E.

If the error voltage at A and B is reversed, the plate of V_2 will be more positive than that of V_1 during the alternations of the voltage across G and H that make both plates positive. The current in the tube V_2 is then greater than that in the tube V_1, and terminal D will now be negative with respect to terminal E.

The circuit therefore delivers a unidirectional voltage at its output terminals D and E, the polarity of which depends on the sense or direction of the error, and the magnitude of which is a direct function of that of the error.

A drawback of the arrangement just described is that it may unduly load the synchros. This is avoided in the circuit arrangement shown in Fig. 8.21, which operates on substantially the same principle of comparing the phase of the error voltage to the fixed phase of the synchro excitation voltage. The error voltage produced by the synchro follow-up is applied to the primary terminals A and B of a high-impedance transformer T. The secondary terminals of this transformer are connected,

through current-limiting resistors r_{g1} and r_{g2}, to the grids of two triodes V_1 and V_2. The filaments or cathodes of these tubes are connected to the center of the transformer secondary winding, while the plates of the tubes, which constitute the output terminals D and E of the circuit, are connected to each other through resistors r_1 and r_2 in series. The junction point H of these resistors and the secondary center tap G of the transformer are connected to the synchro excitation line. The operation of this arrangement is explained in a manner similar to that of the arrangement of Fig. 8.20, and it need not be explained here further.

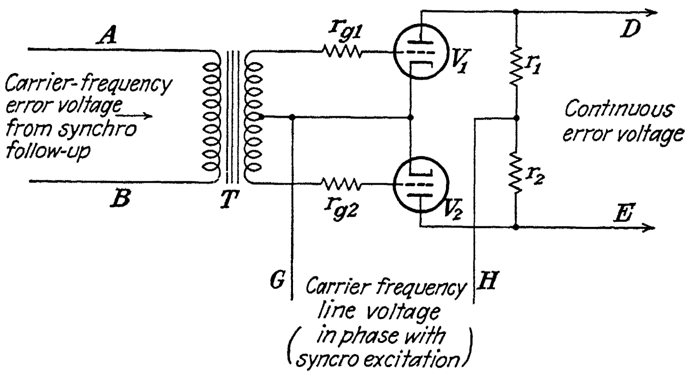

FIG. 8.21.—Demodulating rectifier using triodes in place of diodes.

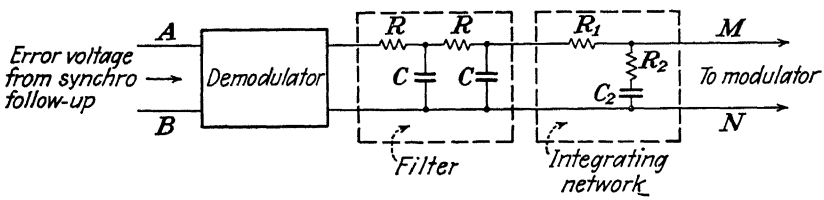

FIG. 8.22.—Demodulating rectifier output filter network.

As mentioned previously, the output voltage of the demodulator, as developed at terminals D and E, is to be applied to the integrating circuit. However, a filter circuit, which may be constituted of series resistors R and shunt capacitors C, as shown in Fig. 8.22, is generally inserted between the demodulator and integrating network, in order to attenuate the higher harmonics of the demodulator output voltage. The filter should be designed to produce only a negligible phase shift of the fundamental error voltage component.

Modulator.—The purpose of the modulator, as stated above, is to transpose to the carrier-frequency level the error-frequency output voltage of the integrating network. A circuit similar to that of the demodulator of Fig. 8.21 may be used, as shown in Fig. 8.23. However,

the input transformer T is here replaced by a resistor r, the center point of which is connected to the cathodes of the two modulator tubes V_3 and V_4. The output voltage of the integrating network is applied to the terminals M, N of this resistor (through an isolating amplifier tube, if necessary, depending on the nature of the integrating circuit). In the absence of any error voltage at the terminals M and N the circuit is balanced, and the carrier-frequency voltage applied at G and H produces

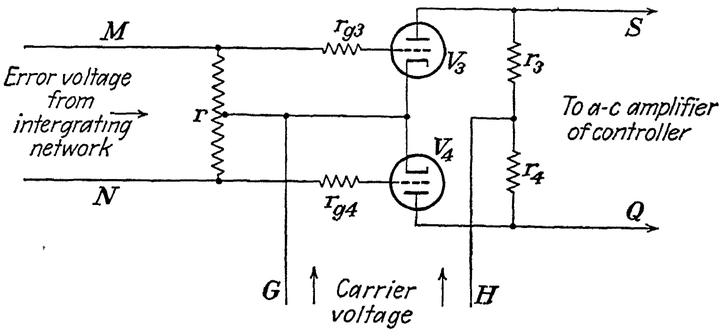

FIG. 8.23.—Modulator for transposing error-frequency output of integrating network to a-c carrier level.

no voltage at the output terminals Q and S. When, though, an error voltage is applied across M, N, the grid voltages of the tubes V_3 and V_4 vary accordingly and in opposite directions. This unbalances the circuit, and a carrier-frequency voltage appears at the terminals Q and S. This voltage is in phase or in phase opposition with the voltage applied at G and H, depending on the polarity of the error voltage. Its amplitude is substantially proportional to the voltage across M, N.

The voltage thus produced at the output terminals Q and S of the modulator can now be applied to an a-c amplifier, and thence to the control windings of an a-c servo motor.

CHAPTER IX

TRANSFER FUNCTION ANALYSIS OF SERVOMECHANISMS

The present chapter constitutes an introductory discussion of the transfer function analysis[1] of servomechanisms. In order to acquaint the reader, in a general way, with this analysis method, its step-by-step development is given in relation to the simple systems previously considered. The study of the logical evolution of this technique will facilitate a full understanding of the use of the stability criterion to which it leads. This stability criterion is particularly convenient for the study of servo control systems more complicated than those discussed in the preceding chapters. A practical example illustrating the application of this powerful technique is worked out in Chap. X, to give to the reader a conception of the processes involved.

In the preceding chapters two methods were used to describe and investigate the performance of servomechanisms. In one of these a step velocity function is applied to the input member of the system, and the input-output error is calculated as a function of time, illustrating the transient response and steady-state behavior of the system. In the other method the system is subjected to sinusoidal motion, and the magnitude and phase of the output displacement, referred to those of the input displacement, are expressed as functions of the frequency. The output-input response (ratio θ_o/θ_i) of the system may then be represented in the form of resonance curves.

This latter method was treated only briefly, and was limited to showing the frequency selective properties of servomechanisms. These properties do not generally furnish a simple criterion for readily ascer-

[1] The concept of the frequency analysis of servomechanisms stems from the frequency analysis (transfer function analysis) of feedback amplifiers first expounded by H. Nyquist, H. W. Bode, and H. S. Black of the Bell Telephone Laboratories, and by A. V. Bedford and G. L. Fredendall of the Radio Corporation of America Laboratories at Princeton, N.J. These methods of analysis were further extended by Donald S. Bond of the Radio Corporation of America, Dr. Albert C. Hall of the Massachusetts Institute of Technology, N. B. Nichols of the Massachusetts Institute of Technology Radiation Laboratory, E. B. Ferrell and LeRoy A. MacColl of the Bell Telephone Laboratories, and others.

A recent book by LeRoy A. MacColl, "Fundamental Theory of Servomechanisms," D. Van Nostrand Company, Inc., New York, 1945, stresses the development of the transfer function concept.

taining the stability of the system considered.[1] This is particularly true in the case of complicated systems having a plurality of degrees of freedom, the operation of which is expressed by differential equations of higher orders.

In order to overcome this limitation, the sinusoidal input method is extended to other functions of the system, the response of which to sinusoidal motion may then be correlated with the stability of the system. Thus, instead of considering the sinusoidal output displacement with respect to the input displacement, as expressed by the ratio θ_o/θ_i, the output displacement may be related to the input-output error (ratio θ_o/θ), or the error may be considered in relation to the input displacement (ratio θ/θ_i). These new functions are generally called *transfer functions*, because they express the transformation that occurs through, or across, a given portion of the servo loop. Of particular usefulness is the output-error function, which represents the response across the controller alone. The use of these transfer functions is based on the analogy that exists between servomechanisms and feedback-connected amplifiers. Transfer function analysis has revealed simple rules to ascertain the stability of any type of servo and to determine the essential parameters of the system.

The present chapter begins with an introduction to the concept of transfer functions. Next, a particular study is made of the output-error transfer function for the same simple servo systems as were discussed before, *viz.*, the viscous-damped, error-rate-damped, and integral control servos. The output vector locus as well as the amplitude-frequency and phase-frequency curves will be derived, in order to familiarize the reader with the concepts involved. Concluding this chapter, an example will be given to show how combinations of damping and control may alter the stability of the system.

A full discussion of this technique, which finds its most effective application in the analysis of more complicated systems, is beyond the scope of this book. For its further study the references quoted in this chapter should be consulted.

Definition of Output-error and Error-input Functions.—It was shown, in previous chapters, that when the input member of a servomechanism is moved back and forth according to a sinusoidal function of time, at some fixed frequency, both the error and the output motion are also sinusoidal functions of time of the same frequency, under steady-state operating conditions. The output displacement, referred to the

[1] For instance, in a simple viscous output damped servomechanism, such as was studied in Chap. IV, reference to Eq. (4.81) shows that the same amplitude-frequency resonance curve will be obtained for a given damping ratio c whether this ratio is positive or negative.

input displacement, was found to be expressed by the complex relation

$$\frac{\theta_o}{\theta_i} = A\epsilon^{j\lambda} \qquad \text{(4.76), (5.45), and (6.48)}$$

in which A is the relative amplitude of the output displacement, and λ the phase angle of the output displacement with respect to the input displacement.

This relation may be represented vectorially, as shown in the left-hand diagram of Fig. 9.1. The vectors θ_i and θ_o of magnitudes equal.

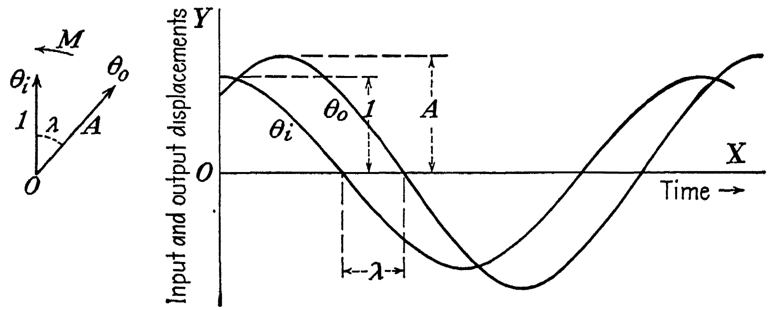

FIG. 9.1.—Relation between sinusoidal input and output displacements.

respectively, to unity and A, represent the input and output displacements. The angle λ between the two vectors is the phase angle between the two displacements. If the entire diagram is rotated counterclockwise, as shown by the arrow M, and at a speed corresponding to the frequency of motion of the servo, the projections of the two vectors along the vertical axis OY represent the instantaneous values of the input and output displacements. These instantaneous values may be plotted against time, measured along the horizontal axis OX, and are then represented by the sine curves shown on the right-hand diagram of Fig. 9.1.

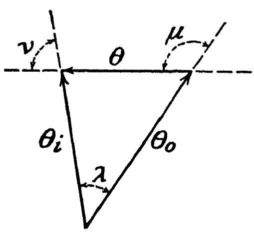

FIG. 9.2.—Vector relations of input and output displacements and of input-output position error.

The vector diagram of Fig. 9.1 is reproduced in Fig. 9.2. The input-output position error θ, defined as the difference

$$\theta = \theta_i - \theta_o \tag{9.1}$$

is represented by the vector θ. When subtracted vectorially from the input vector θ_i, this error vector θ produces the output vector θ_o. The output displacement lags behind the error by an angle μ. From these geometrical considerations, it is then seen that, just as the output displacement was expressed in relation to the input

displacement by the equation

$$\frac{\theta_o}{\theta_i} = A\epsilon^{j\lambda} \tag{9.2}$$

it can also be expressed relative to the error by the equation

$$\frac{\theta_o}{\theta} = B\epsilon^{j\mu}. \tag{9.3}$$

Similarly, the error can be expressed in its relation to the input displacement by the equation

$$\frac{\theta}{\theta_i} = C\epsilon^{j\nu}. \tag{9.4}$$

Taking into account Eq. (9.1), the last three expressions can be combined to obtain some further relations, which will be applied later.[1] Thus,

$$B\epsilon^{j\mu} = \frac{\theta_o}{\theta} = \frac{\theta_i - \theta}{\theta} = \frac{\theta_i}{\theta} - 1 \tag{9.5}$$

from which

$$\frac{\theta_i}{\theta} = 1 + B\epsilon^{j\mu} \qquad \text{or} \qquad \frac{\theta}{\theta_i} = \frac{1}{1 + B\epsilon^{j\mu}}. \tag{9.6}$$

Thus, knowing θ_o/θ, it becomes possible to calculate θ/θ_i, and it follows that

$$A\epsilon^{j\lambda} = \frac{\theta_o}{\theta_i} = \frac{\theta_o}{\theta}\frac{\theta}{\theta_i} = \frac{B\epsilon^{j\mu}}{1 + B\epsilon^{j\mu}}. \tag{9.7}$$

This relation, in turn, allows the output-error function θ_o/θ to be expressed in terms of the output-input function θ_o/θ_i.

$$B\epsilon^{j\mu} = \frac{A\epsilon^{j\lambda}}{1 - A\epsilon^{j\lambda}}. \tag{9.7a}$$

Similarly, it is found at once that

$$C\epsilon^{j\nu} = \frac{1}{1 + B\epsilon^{j\mu}}. \tag{9.8}$$

It is to be noted that the above equations are complex expressions and that the operations indicated must be performed vectorially.

Comparison with Feedback Amplifier.—The preceding relations of a servomechanism are analogous to those which represent the operation of a feedback-connected amplifier system, as used widely in radio and com-

[1] It is the output-error function, Eq. (9.3), which will be studied at greater length in the paragraphs that follow.

munication apparatus. A simple amplifier, as shown in Fig. 9.3, when subjected to a sinusoidal voltage E_i applied to its input terminals A and B, develops a sinusoidal voltage E_o of same frequency but different amplitude at its output terminals C and D. The output-input voltage ratio $m = E_o/E_i$ is called the gain of the amplifier.

A feedback-amplifier system, illustrated in Fig. 9.4, differs from the simple amplifier arrangement in that the output voltage E_o, in addition

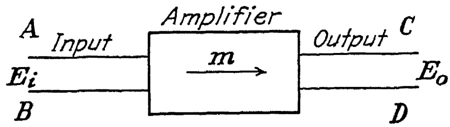

Fig. 9.3.—Simple amplifier.

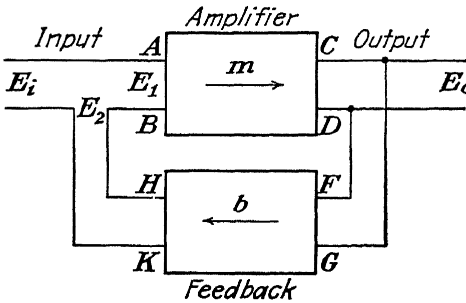

Fig. 9.4.—Feedback amplifier.

to being applied to an external load (not shown in the figure), is also impressed on the input terminals F and G of an auxiliary amplifier. Designating the *gain* of this auxiliary amplifier by b, the voltage $E_2 = bE_o$ appearing at the output terminals K and H is applied to the input circuit of the main amplifier, in series with the input voltage E_i of the system.

The voltage E_1 actually applied to the terminals A and B is thus equal to

$$E_1 = E_i + E_2, \tag{9.9}$$

and since

$$\left\{\begin{aligned} m &= \frac{E_o}{E_1} \qquad (9.10)\\ b &= \frac{E_2}{E_o} \qquad (9.11)\end{aligned}\right.$$

it follows that

$$E_2 = mbE_1 \tag{9.12}$$

and

$$E_1 = E_i + mbE_1 \tag{9.13}$$

or

$$E_1 = \frac{E_i}{1 - mb}. \tag{9.14}$$

Substituting this in Eq. (9.10),

$$\frac{E_o}{m} = \frac{E_i}{1 - mb} \tag{9.15}$$

or

$$\frac{E_o}{E_i} = \frac{m}{1 - mb}. \tag{9.16}$$

Comparing this last relation with Eq. (9.7),

$$A\epsilon^{j\lambda} = \frac{\theta_o}{\theta_i} = \frac{B\epsilon^{j\mu}}{1 + B\epsilon^{j\mu}} \tag{9.7}$$

it is seen that a servo system may be likened to a feedback amplifier having a complex forward gain

$$m = B\epsilon^{j\mu} \tag{9.17}$$

and a negative feedback gain equal to

$$b = -1. \tag{9.18}$$

Thus, the same methods used to study the properties of feedback amplifiers may be applied to a discussion of corresponding properties of servomechanisms.

One of these methods[1] consists in constructing a vector diagram similar to that of Fig. 9.2 for various values of the frequency. The error vector being taken as reference, both as to phase and amplitude, the locus of the output displacement vector will then furnish complete information on the stability conditions of the system.

Another method[2] brings in the frequency explicitly as the independent variable, as a function of which the amplitude and phase of the output vector are expressed, relative to the input-output position error. These relations, represented graphically, result in curves that take the place of the usual frequency response curves, and allow a complete determination of the parameters and operating characteristics of the servomechanism.

Before applying these methods to the various types of servo control systems discussed in previous chapters, a brief outline of a possible experimental procedure will be given to supply a concrete explanation of the problem treated in the paragraphs that follow.

Experimental Procedure.—The experimental procedures described here permit actual measurements of the variables to be made, which will enter in the discussion of the output-error transfer function of the system.

[1] See NYQUIST, H., Regeneration Theory, *Bell System Tech. J.*, vol. XI, pp. 126–147, January, 1932.

[2] See BODE, H. W., Relations between Attenuation and Phase in Feedback Amplifier Design, *Bell System Tech. J.*, vol. XIX, pp. 421–454, July, 1940.

Figure 9.5 shows a servo control system similar to those studied in previous chapters. The controller comprises an a-c amplifier and motor that drives the output load. The position of the load is transmitted through a synchro generator, excited from a 60-cycle a-c supply line, to a synchro control transformer. The rotor of this differential synchro is driven by the input member of the system. The output voltage of the differential is proportional to the input-output position error and is applied to the input terminals of the controller amplifier. Although a

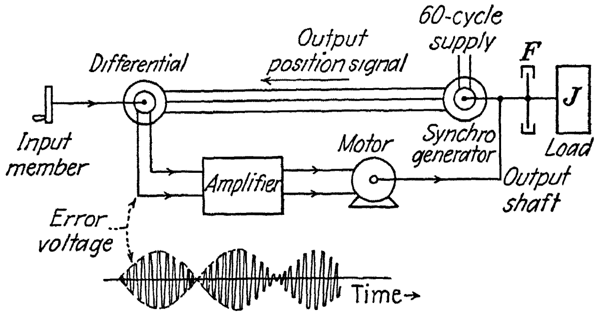

FIG. 9.5.—Servomechanism with synchro follow-up link.

friction damping device is shown in the diagram of Fig. 9.5, the procedure to be described applies equally well to any of the other types discussed before, such as the error-rate stabilized system, for example.

If the input member of the system is displaced alternately back and forth, the error voltage, of 60-cycle frequency, is amplitude modulated at the frequency of the input motion.[1] This voltage, after amplification, is applied to the motor, which imparts a corresponding alternately reversed motion to the output load. The alternating output motion, of the same frequency as the input motion, is transmitted back through the synchro generator, and modifies the error signal produced in the differential synchro. This, in turn, alters the applied motor voltage, and the motor torque and speed, in such manner as to maintain the position correspondence between the input and output members of the system, within the error limits afforded by the design.

In order to study the performance of the system, a possible operating procedure consists in measuring the amplitude, phase, and frequency of the input and output motions and of the error voltage or signal. Relations among these three quantities can then be established and discussed in terms of the parameters of the system.

Another procedure for establishing these relations, as described below, eliminates the reaction of the output member displacement upon the error signal during the measurements. The measured quantities are

[1] This is actually suppressed-carrier amplitude modulation.

the same as though the reaction existed, but the measurements are made under conditions that correlate them more directly with the proposed theoretical study of the servo system considered, as outlined in this chapter. As shown in Fig. 9.6, this is done by opening the circuit through which the error signal is transmitted to the amplifier, and then by feeding into the amplifier an externally produced, artificial error voltage equivalent to the error voltage actually produced in the system under normal operating conditions. This artificial error voltage, like the original error voltage, is an amplitude-modulated 60-cycle voltage. An amplitude modulation of 1 volt corresponds to an input-output position error of 1 deg. in the actual synchro system.

It may be readily understood that if, in the experimental arrangement shown in Fig. 9.6, the voltage applied to the amplifier is an unmodulated

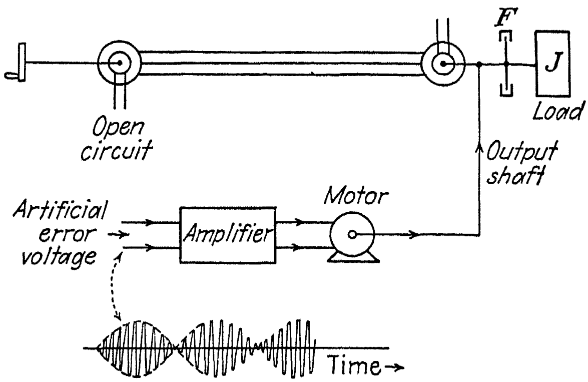

FIG. 9.6.—Experimental measurement of output-error function.

alternating voltage, the motor will run continuously in a fixed direction. This corresponds to a constant input-output error in the actual system of Fig. 9.5. However, if the artificial error voltage in Fig. 9.6 is alternately reversed, at some given frequency, the motor will also reverse alternately at this same frequency. During every alternation it will rotate the load by a certain number of turns. If the error voltage is reversed more frequently, the motor reversals will be correspondingly more frequent, and the output displacement (measured in number of turns, or radians, or degrees) during each alternation is generally reduced. Thus, the number of turns or degrees of rotation of the output member decreases as the error frequency increases. This amplitude of the output displacement, for an error of constant amplitude and increasing frequency, does not, however, vary exactly in inverse proportion to this frequency. One reason is that the inertia of the system (motor, gears, load, etc.) prevents the output member from accelerating sufficiently fast, as the error frequency is increased. It should be noted for similar reasons that

the *phase* of the output displacement varies with respect to the error voltage by an angle that depends on the frequency.

Thus, by opening the servo loop and by applying a suitably modulated error signal to the controller, it is possible to investigate the effect of the operating frequency on the amplitude and phase of the output displacement. Since these effects are also functions of the system parameters, the characteristics of the system can be determined as they will appear when the system loop is again closed, and the mechanism operated under its normal conditions.

SERVOMECHANISMS WITH VISCOUS OUTPUT DAMPING

Expression of Output Displacement as a Function of Input-output Error.—The relation between the output displacement of a servomechanism and the input-output error will be derived in terms of the system parameters. As will be shown later, this relation leads to simple methods of measuring and discussing the operating characteristics of the servo.

The equation of motion of a servomechanism with viscous output damping was given in Chap. IV as

$$K\theta = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt}. \tag{4.1}$$

If the error θ is a sinusoidal function of time of unit amplitude

$$\theta = \epsilon^{j\omega t} \tag{9.19}$$

the output displacement θ_o, according to Eq. (9.3), is equal to

$$\theta_o = B\epsilon^{j(\omega t+\mu)}. \tag{9.20}$$

The first and second time derivatives of this expression are, respectively,

$$\frac{d\theta_o}{dt} = j\omega B\epsilon^{j(\omega t+\mu)} \tag{9.21}$$

$$\frac{d^2\theta_o}{dt^2} = -\omega^2 B\epsilon^{j(\omega t+\mu)}. \tag{9.22}$$

Substituting the Eqs. (9.19), (9.21), and (9.22) in Eq. (4.1), this becomes

$$K\epsilon^{j\omega t} = -\omega^2 JB\epsilon^{j(\omega t+\mu)} + j\omega FB\epsilon^{j(\omega t+\mu)} \tag{9.23}$$

or, after dividing through by $\epsilon^{j\omega t}$ and collecting terms,

$$B\epsilon^{j\mu} = \frac{K}{-\omega^2 J + j\omega F}. \tag{9.24}$$

This is the complex expression of the output amplitude and phase response of the system, referred to the input-output error, in terms of the system parameters.

In passing, it may be noted that this expression can be checked by writing it in Eq. (9.7).

$$A\epsilon^{j\lambda} = \frac{B\epsilon^{j\mu}}{1 + B\epsilon^{j\mu}} \tag{9.7}$$

which then becomes

$$A\epsilon^{j\lambda} = \frac{\dfrac{K}{-\omega^2 J + j\omega F}}{1 + \dfrac{K}{-\omega^2 J + j\omega F}} = \frac{K}{-\omega^2 J + j\omega F + K}.$$

This is the same as Eq. (4.76) of Chap. IV.

The expression, Eq. (9.24), may be written in simpler form in terms of the damping ratio c and relative frequency $d = \omega/\omega_n$ used in the preceding chapters. Thus, dividing both numerator and denominator by K in Eq. (9.24), there results

$$B\epsilon^{j\mu} = \frac{1}{-\omega^2(J/K) + j\omega(F/K)}. \tag{9.25}$$

Substituting in Eq. (9.25) the relations, Eqs. (4.58) and (4.61), of Chap. IV, this expression becomes

$$B\epsilon^{j\mu} = \frac{1}{-\dfrac{\omega^2}{\omega_n{}^2} + 2jc\dfrac{\omega}{\omega_n}} \tag{9.26}$$

or finally

$$B\epsilon^{j\mu} = \frac{1}{-d^2 + 2jcd}. \tag{9.27}$$

This complex expression of the output displacement represents a sinusoidal function of time of amplitude B and phase angle μ (relative to the error) giving

$$\begin{cases} B = \dfrac{1}{\sqrt{d^4 + 4c^2d^2}} & (9.28) \\[2ex] \mu = \tan^{-1}\dfrac{2c}{d} & (9.29) \end{cases}$$

These relations, Eqs. (9.27), (9.28), and (9.29), could have been obtained also by substituting in Eq. (9.7*a*) the Eq. (4.80) given in Chap. IV for the output-input relation. This procedure will be applied in later parts of this chapter, when other types of servomechanisms are discussed.

Locus of Output Displacement Vector.—Since the error, input, and output displacements are sinusoidal functions of time, they can be represented vectorially in the manner shown in Fig. 9.2. The diagram is fully

determined, and can actually be constructed if, as is done here, the error vector θ is taken as the reference (unit length, and zero phase angle), while the length and phase angle of the output displacement vector θ_o are known from the above relations Eqs. (9.28) and (9.29).

In passing, it might be noted that the form of the vector diagram of Fig. 9.2 is not quite conventional. In order to better illustrate the relation between the two vectors θ_o and θ, this diagram can be changed into that of Fig. 9.7 by sliding the vector θ_o along its own direction until its origin coincides with that of the vector θ. Then, the latter is taken as the reference vector (that is to say, a vector whose length is equal to unity, and from whose direction the direction of other vectors is measured), and the entire diagram is rotated by 180 degrees around the point of origin of the two vectors θ and θ_o. This is shown

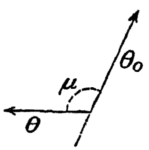

Fig. 9.7.—Output displacement and error vectors.

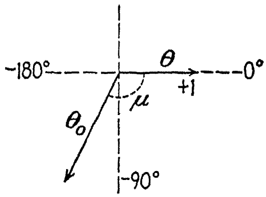

Fig. 9.8.—Output displacement and error vectors with error vector as reference.

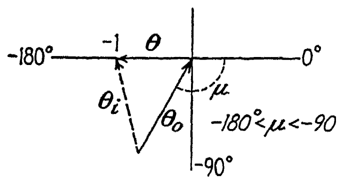

Fig. 9.9.—Output displacement and error vectors, conventional diagram.

in Fig. 9.8. The reference vector θ will then point to the right, and extend to the point of abscissa $+1$ along the horizontal axis.

The angle μ between the two vectors, measured positively in the counterclockwise direction in accordance with standard practice, is then as defined by the above relation, Eq. (9.29). In this relation, the factor c is the damping ratio of the servo system, and is always positive under the conditions considered here. Since the frequency ratio $d = \omega/\omega_n$ is also positive, it is seen from Eq. (9.29) that the angle μ is always comprised between 180 and 270 deg. or, in equivalent manner, between -180 and -90 deg.

From the diagram of Fig. 9.8 it is now possible to return to that of Fig. 9.2, as shown in Fig. 9.9, by reversing the two vectors θ and θ_o of Fig. 9.8. The error vector θ then extends to the abscissa -1 on the horizontal axis, while the length B and angular position μ of the output vector θ_o are given by Eqs. (9.28) and (9.29), respectively. The input vector θ_i is shown in dotted line for completeness of the diagram.

Keeping the damping ratio c to a fixed value, determined by the constants of the servomechanism considered, Eqs. (9.28) and (9.29) show that the output displacement amplitude B (in terms of the error ampli-

tude taken as unity), as well as the phase angle μ, decrease[1] for increasing values of the relative frequency d. Thus, keeping the error amplitude equal to unity and changing the frequency d will cause the origin of the output vector θ_o (and also of the input vector θ_i) to be displaced on the diagram. The resulting curve, or locus, is shown on the right in Fig. 9.10, as calculated from Eqs. (9.28) and (9.29) for different fixed values of c.

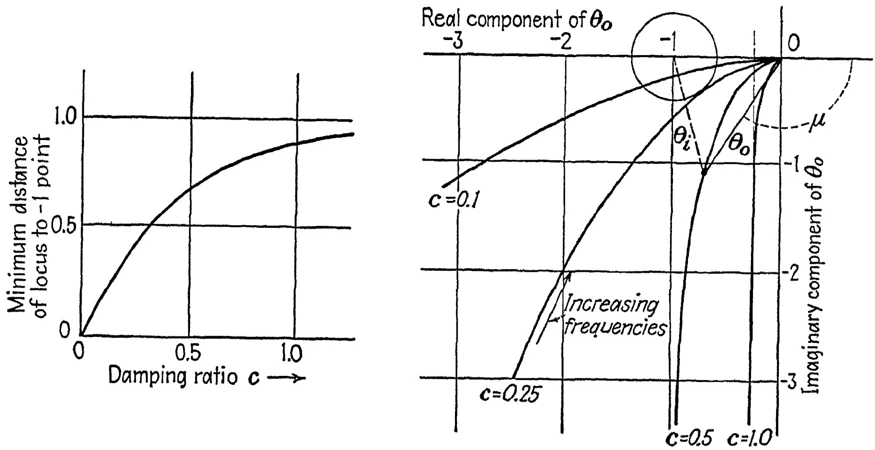

FIG. 9.10.—Output vector locus of viscous-damped servomechanism.

For large values of the relative frequency d the output vector is small. Its magnitude increases as the frequency d decreases, but its phase angle at first varies slowly with the frequency. The output vector locus then falls away from the horizontal axis more and more rapidly as d is made smaller and smaller. It finally approaches a vertical asymptote, which cuts the horizontal axis at a point of ordinate equal to $-1/4c^2$. Thus, the curve comes closer to the -1 point of the horizontal axis, as the value of the damping ratio c decreases.

The value of c may therefore be indicated by the proximity of the curve to this -1 point, which, in turn, may be measured by the radius of a circle having the -1 point as its center and drawn tangent to the curve. It was shown in Chap. IV that the smaller the value of c, the greater will be the tendency of the system to oscillate, and, as illustrated here, the radius of the circle just defined decreases with the value of c. Therefore, this radius may, in the present instance, serve as an indication of the degree of stability of the system, in place of the value of the damping ratio c. The left-hand diagram of Fig. 9.10 shows the relation between the values of c and of the radius of the tangent circle.

[1] The angle being negative, the statement that the angle decreases implies that its absolute value (or magnitude) increases. The quadrant in which the output vector lies is determined from Eq. (9.27).

The reason for referring the curve to the -1 point of the horizontal axis is that if the curve should happen to cut the horizontal axis at this point (which, however, does not occur in this type of servo unless c equals zero), the input vector θ_i would be reduced to zero. In other words, no input motion would be required, and the system, after having once been started, would oscillate continuously without any external input stimulus, *i.e.*, the system would be unstable.

The position of the output vector locus with respect to the -1 point of the horizontal axis thus indicates the stability of the system. This feature will be brought out later in greater detail, when it will also be seen that substantially the same circle, having the -1 point as its center, is tangent to the output vector locus of all systems having the same effective damping ratio value. This obtains irrespective of the type of damping used, such as viscous output damping or error-rate damping, for example.

Amplitude-frequency Response of Output-error Function.—While the vector representation just described shows the relationships of the various displacements involved, it does not explicitly bring out the dependence of the variables on the operating frequency of the system. This dependence is more properly illustrated by a curve representing the magnitude or the phase of the displacement against the frequency. Such curves were given in previous chapters to show the dependence, on frequency, of the output displacement expressed in terms of the input displacement.

In the present instance, it is the output displacement expressed in terms of the input-output error that is to be represented as a function of the operating frequency. Both the amplitude and phase of the output displacement can readily be measured, for any desired error signal, by applying the experimental procedure described above in connection with Figs. 9.5 and 9.6. The same relations, from which quantitative results affecting the design and performance of the servo system may be derived, are determined mathematically in the following discussion.

When a viscous-damped servomechanism is operated under such conditions that the input-output position error is a sinusoidal function of time of unit amplitude, the output displacement was shown to have an amplitude equal to

$$B = \frac{1}{\sqrt{d^4 + 4c^2d^2}}. \tag{9.28}$$

In this equation, the damping ratio c is fixed for the particular servomechanism considered, while the relative frequency d and output displacement amplitude B depend on the operating conditions of the system. Thus, if d is made to tend toward zero, B tends toward the value

$$B = \frac{1}{\sqrt{4c^2d^2}} = \frac{1}{2cd}. \tag{9.30}$$

Conversely, if d is made increasingly larger, B tends toward the value

$$B = \frac{1}{\sqrt{d^4}} = \frac{1}{d^2}. \tag{9.31}$$

These two equations represent the asymptotes of the amplitude-frequency curve of the system, as expressed by Eq. (9.28).

In these equations B is the ratio of the amplitude of the output displacement to that of the error, while d is the ratio of the operating frequency to the natural oscillation frequency of the system. As in communication engineering, it is convenient here, also, to deal with the logarithms of such ratios, rather than with the ratios themselves. Thus, the last two equations may be written in the form

$$\log B = -\log 2cd = -\log d - \log 2c, \tag{9.32}$$

and

$$\log B = -2 \log d. \tag{9.33}$$

Both these relations show log B to be proportional to log d. They can, therefore, be represented graphically by straight lines, shown as S and T, respectively, in Fig. 9.11. These lines are asymptotes of the response

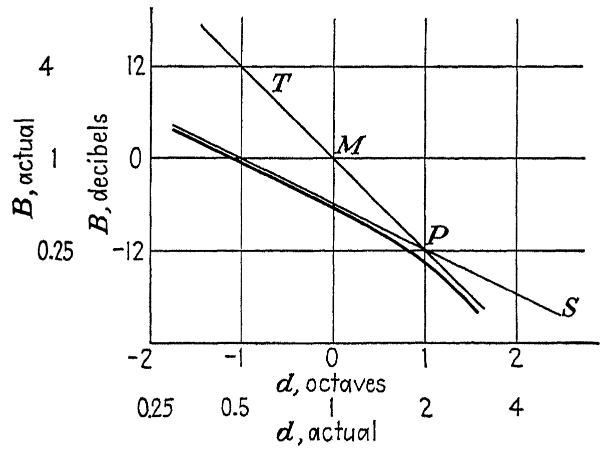

FIG. 9.11.—Output displacement amplitude (relative to error amplitude) as function of frequency in a viscous-damped servomechanism.

curve of the system, representing Eq. (9.28), and drawn in heavy line in the figure. Actually, the scales along the axes of Fig. 9.11 are not graduated in values of log B and log d directly, but in decibels and

octaves, respectively, which are directly proportional to the logarithms themselves.[1]

These units being used, Eq. (9.32) shows that the output displacement amplitude B increases by 6 db when the frequency d decreases by 1 octave at frequencies substantially smaller than the natural frequency of the system. Similarly, Eq. (9.33) shows that B increases by 12 db when d decreases by 1 octave at frequencies substantially greater than the natural frequency.

In the limiting case, referring back to Eq. (9.28), if the servo system is undamped ($c = 0$), the expression for B reduces to

$$B = \frac{1}{\sqrt{d^4}} = \frac{1}{d^2}$$

which is exactly the same as Eq. (9.31), and is represented by the straight line T in Fig. 9.11. This equation shows that

$$B = \frac{\theta_o}{\theta} = 1 \qquad \text{when} \qquad d = \frac{\omega}{\omega_n} = 1. \tag{9.34}$$

Expressed in words, if an error signal corresponding, for example, to a sinusoidal error of 10 deg. amplitude is applied to the controller of the servo system illustrated in Fig. 9.6 and the error frequency is adjusted to a value for which the displacement amplitude of the output member is also 10 deg., this frequency value will be equal to the natural frequency of the system, if the servo is undamped.

The relation also shows that the straight line T of Fig. 9.11 passes through the point $B = 0$ db, $d = 0$ octave, shown at M in the graph.

Finally, the two lines S and T of Fig. 9.11 cross at a point P for which the values of B, as given by the Eqs. (9.30) and (9.31), are equal. Thus,

$$\frac{1}{2cd} = \frac{1}{d^2}, \tag{9.35}$$

from which

$$\frac{\omega_p}{\omega_n} = d = 2c; \tag{9.36}$$

or

$$c = \frac{\omega_p}{2\omega_n}. \tag{9.37}$$

In order to determine this point P experimentally, the output displacement amplitude is measured in terms of the applied error signal

[1] It is recalled that the value of the amplitude ratio B expressed in decibels is equal to $20 \log_{10} B$. On the other hand, if d is the ratio of two frequencies, this ratio, expressed in octaves, is equal to $\log_2 d$.

amplitude for different values of the error frequency. The values obtained are plotted, and a curve is drawn, as shown in heavy line in Fig. 9.11. The two asymptotes S and T can then be drawn as tangents to the curve, with slopes of 6 and 12 db per octave variation of d, and their crossing point P is thereby obtained. The frequency value ω_p corresponding to this point, together with the value ω_n obtained from the intersection of the line T with the $B = 0$ db level, as described before, then allows the damping ratio c of the system to be calculated, according to Eq. (9.37).

Having found ω_n and c as just explained, the acceleration and velocity figures of merit of the system can be computed, by applying the relations, Eqs. (4.91) and (4.89) of Chap. IV.

$$M_a = \frac{K}{J} = \omega_n^2 \tag{4.91}$$

$$M_\omega = \frac{K}{F} = \frac{\omega_n}{2c}. \tag{4.89}$$

Moreover, Eq. (9.37) can be written

$$\omega_p = 2c\omega_n = 2c\sqrt{\frac{K}{J}} = 2c\,\frac{\sqrt{KJ}}{J}$$

which, from Eq. (4.58) of Chap. IV, gives

$$\omega_p = \frac{F}{J}. \tag{9.38}$$

Knowing ω_n, c, and one of the three constants F, K, and J of the servo system under test, it is then possible to calculate the other constants from Eqs. (4.91), (4.89), and (9.38).

Problem.—In a viscous-damped servomechanism, such as that shown in Fig. 9.5, the 60-cycle synchro repeater follow-up link supplies to the controller a voltage of 1 volt per deg. of input-output position error. The loop being opened and the system rearranged as shown in Fig. 9.6, a modulated 60-cycle voltage of 1-volt amplitude is applied to the controller. The measurements listed below having been obtained, calculate the constants of the servomechanism.

Error Frequency, Cycles per Sec.	Output Displacement Amplitude, Deg.
0.5	16
1	8
8	1/4
16	1/16

Solution: Since the natural frequency ω_n of the system is unknown, the actual error frequency values are plotted along a horizontal axis, so graduated that equal lengths correspond to frequency differences of one octave (Fig. 9.12).

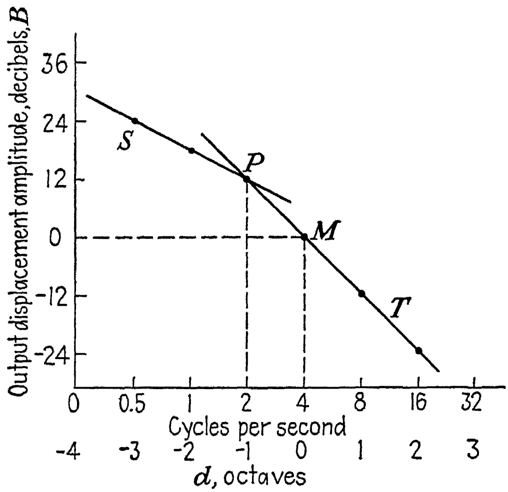

FIG. 9.12.—Experimental determination of servomechanism parameters.

On the other hand, the output displacement amplitude is plotted along the vertical axis, after having been converted into decibels.

Recalling that the applied error voltage is equivalent to an input-output error of 1 deg., these values are

Output Displacement Amplitude Relative to Error	Db.
16.1	24
8:1	18
¼:1	−12
1/16:1	−24

The first two measurements obviously were made at frequencies well below the natural frequency of the system, while the last two measurements correspond to frequencies well above this. Therefore, each of these two pairs of measurements can be used to determine one of the asymptotes of the response curve of the system. These are drawn as straight lines S and T on the diagram, remembering that their slopes are, respectively, 6 and 12 db per octave.

The point M of the line T having an ordinate of 0 db corresponds to an error frequency of 4 cycles per sec., which therefore is the natural frequency ω_n of the system. This point also constitutes the *zero* point of the octave scale on the horizontal axis of relative frequencies.

On the other hand, the crossing point P of the lines S and T is one octave below the point M. It thus corresponds to an actual frequency ω_p that is equal to one-half ω_n.

$$\omega_p = \frac{\omega_n}{2} = \frac{4}{2} = 2 \text{ cycles per sec.}$$

Writing this in Eq. (9.37), the damping ratio c of the servo is

$$c = \frac{\omega_p}{2\omega_n} = \frac{1}{4}.$$

With ω_p, ω_n, and c known, the other constants of the system can readily be calculated from the preceding relations, Eqs. (4.94), (4.95), and (9.38). In particular, the velocity figure of merit, as defined in Eq. (4.95) of Chap. IV, is equal to

$$M_v = \frac{\omega_n}{2c} = \frac{4 \times 2\pi}{2 \times \frac{1}{4}} = 50,$$

in which the natural frequency of 4 cycles per sec. is expressed in radians per second. Since, by definition,

$$M_v = \frac{\omega_1}{\theta},$$

as explained in Chap. IV, the steady-state error θ for an input speed ω_1 of 10 r.p.m. (or 1 radian per sec.), after the servo loop has been restored as shown in Fig. 9.5., is given by

$$\begin{aligned} \theta &= \frac{\omega_1}{M_v} = \frac{1}{50} \text{ radian} \\ &= \frac{57.3}{50} \text{ deg.} = \text{about 1 deg.} \end{aligned}$$

Phase-frequency Response of Output-error Function.—It was shown in relation to Eqs. (9.28) and (9.34) that in an undamped servomechanism (damping ratio c equal to zero) the error and output displacement have equal amplitudes when the error actuating the controller has a frequency equal to the natural frequency of the system. This is expressed mathematically as

$$\begin{cases} B = \dfrac{\theta_o}{\theta} = 1 \\ \text{when } d = 1 \qquad \text{and} \qquad c = 0. \end{cases}$$

Equation (9.29) shows that the phase angle of the output displacement with respect to the error is then equal to

$$\mu = \tan^{-1} 0 = -180 \text{ deg.}$$

The corresponding vector relationship between the error and output displacement is illustrated in the left-hand diagram of Fig. 9.13. By comparison with the vector diagrams given previously (see Fig. 9.2, for example), it is seen that the input vector θ_i joining the origin of the output vector θ_o to the end of the error vector θ is, in the present instance, reduced to zero. In other words, the system, having been set in motion, oscillates continuously at its natural frequency without any displacement of the input member being required to sustain the motion.

Referring now to the output vector locus shown in Fig. 9.10 and reproduced in the right-hand diagram of Fig. 9.13, it is seen that when the servomechanism is damped ($c \neq 0$), an output displacement of the same amplitude as the error is obtained with a phase angle comprised between -180 and -90 deg. As the damping ratio c is increased, the phase angle departs more and more from -180 deg. and tends increasingly toward the value of -90 deg. Although this is not immediately apparent from the diagram, as the damping ratio c is increased, the error frequency at which the output amplitude B is equal to unity decreases continuously from the natural frequency value down.

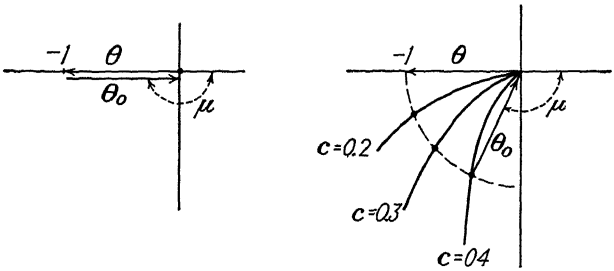

FIG. 9.13.—Effect of damping ratio on output displacement vector phase angle.

Of course, the smaller the damping ratio c, the less stable is the servo, and the greater is its tendency to oscillate. It was found in Chap. IV that in practice the value of c is seldom made smaller than 0.2, which corresponds approximately to a phase angle of -158 deg. at $d = 1$. Below this value of c, and therefore above this value of the phase angle, the transient oscillations of the system are greater than can generally be tolerated. Thus, the phase angle, when $\omega = \omega_n$, just as well as the damping ratio, can serve to characterize the stability of the servomechanism.

Equation (9.27) and the output vector locus diagram are compact representations of the output displacement, as referred to the error. They show simultaneously the magnitude and phase angle of the output vector. These two quantities can, however, be represented separately as functions of the frequency, by applying Eqs. (9.28) and (9.29), respectively. This was done previously in relation to the output displacement amplitude, and will be done presently for the phase angle.

For any given fixed value of the damping ratio c Eq. (9.29) allows the phase angle μ to be calculated as a function of the relative frequency d. Thus, the phase angle approaches -90 deg. when the frequency is made to tend toward zero. It approaches -180 deg. when the frequency is increased indefinitely. This is illustrated by the curve G in Fig. 9.14, where the values of the phase angle, in degrees, are represented as functions of the relative frequency, expressed in octaves.

Corresponding values of the output displacement amplitude B, expressed in decibels as before, and derived from Eq. (9.28), are repre-

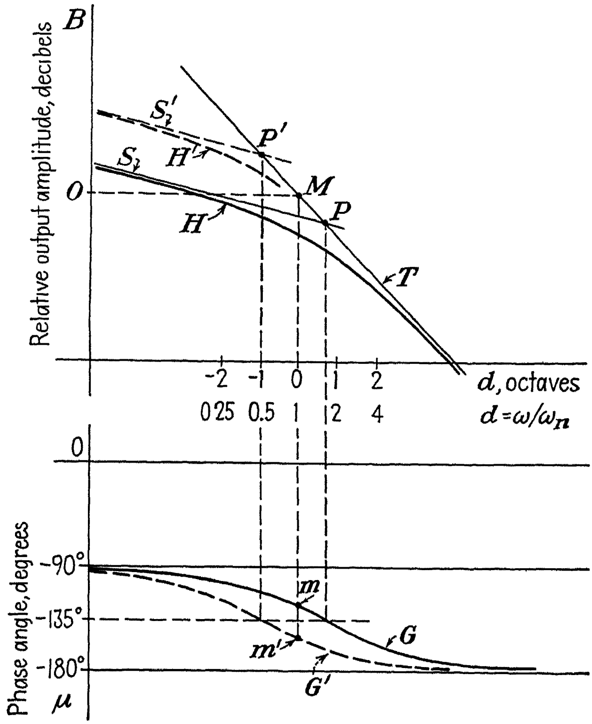

FIG. 9.14.—Output vector amplitude and phase angle as functions of frequency in a viscous-damped servomechanism.

sented by the curve H of Fig. 9.14. The crossing point P of the two asymptotes S and T was shown to correspond to a frequency

$$d_p = 2c. \qquad \text{(see Eq. 9.36)}$$

At this same frequency, the phase angle, according to Eq. (9.29), is equal to

$$\mu = \tan^{-1}\frac{2c}{d} = \tan^{-1} 1 = -135 \text{ deg.}$$

Now let M be the point of the 12-db slope asymptote T for which the output displacement amplitude is equal to the error (relative value of $B = 1$, or amplitude ratio = 0 db). As explained before (see Fig. 9.11), this corresponds to a frequency equal to the natural frequency of the system, for which $d = 1$ (or zero octave). With the value of c chosen to draw the curves G and H of the figure, the point P lies to the right of point M, and the corresponding frequency d_p is above the *zero octave*

reference frequency. For the natural frequency of the system the phase angle of the output displacement, as shown by point m of the curve G, is then located between -135 and -90 deg., and the system has a good margin of stability.

Suppose now that the system has a damping ratio c' smaller than the value c just considered. According to Eq. (9.36), the frequency value d'_p for which the phase angle is -135 deg. will be correspondingly smaller than the previous value d_p. The curve G is then shifted toward the left, as shown, for example, by the curve G'. At the same time, the curve H, asymptote S, and point P are shifted upward as shown in H', S', and P'. At the natural oscillation frequency of the system the phase angle now corresponds to the point m' of the curve G', and thus is located between -135 and -180 deg. If the value c' is so small as to make the phase angle approach -180 deg. (or, as explained before, exceed -158 deg.), the system becomes too unstable for the practical applications usually encountered. This corresponds to a position of the point P' approximately one octave below the natural frequency of the system.[1] Thus, mere inspection of the amplitude-frequency curve shows whether the system is or is not of practically sufficient stability, by observing the position of the crossing point P of the two asymptotes S and T of the curve.

[1] At the frequency corresponding to the crossing point P of the two asymptotes, the phase angle is

$$\mu = -180 + 45 = -135 \text{ deg.}$$

and

$$\tan \mu = 1 = \frac{2c}{d}.$$

If the damping c is such that the crossing point corresponds to the natural frequency of the system ($d = 1$), then

$$1 = \frac{2c}{1} \qquad \text{or} \qquad c = 0.5.$$

Now let the damping be reduced to the lowest acceptable value c', for which the phase angle at the frequency $d = 1$ is

$$\mu = -180 + 22 = -158 \text{ deg.}$$

and

$$\tan \mu = 0.404.$$

then

$$0.404 = \frac{2c'}{d} = \frac{2c'}{1} \qquad \text{from which} \qquad c' = 0.202.$$

The crossing point of the two asymptotes is thereby shifted to a frequency d' such that

$$\frac{2c'}{d'} = 1 \qquad \text{or} \qquad d' = 2c' = 0.404$$

or slightly more than one octave below the natural frequency of the system.

It should be noted that if there were no damping ($c = 0$), the point P and asymptote S would be shifted infinitely far upward. The amplitude curve would then simply be the straight line T, with a negative slope of 12 db per octave. The phase angle would be -180 deg., and the system would be unstable. Introduction of damping into the system reduces the slope of the amplitude curve to 6 db per octave at the lower frequencies, and the phase angle then tends toward the value of -90 deg. The correlation of the slope of the amplitude curve and the phase angle (6 db and -90 deg. on the one hand, and 12 db and -180 deg. on the other) shows that, in order to stabilize the system, sufficiently high damping must be introduced to bring the crossing point of the asymptotes of the amplitude curve close to the natural frequency of the system (not more than one octave below).

SERVOMECHANISMS WITH ERROR-RATE DAMPING

Expression of Output Displacement as a Function of Input-output Error.—The relation between the output displacement and the error in an error-rate stabilized servomechanism can be derived in a manner similar to that followed in the case of a servo with viscous output damping: the error being a sinusoidal function of time, as in Eq. (9.19), this error and the corresponding output displacement Eq. (9.20) are written in the equation of motion of the servo, given as Eq. (5.2) in Chap. V, from which the output-error transfer function is then obtained.

The same result may be arrived at by substituting in Eq. (9.7*a*) the expression of the output-input function $A\epsilon^{j\lambda}$, as found in the relation, Eq. (5.52), of Chap. V.

$$\left\{\begin{aligned} A\epsilon^{j\lambda} &= \frac{1 + j2cd}{1 - d^2 + j2cd} && (5.52) \\ B\epsilon^{j\mu} &= \frac{A\epsilon^{j\lambda}}{1 - A\epsilon^{j\lambda}} && (9.7a) \end{aligned}\right.$$

The output displacement is then expressed as a function of the frequency of a sinusoidal error of unit amplitude by the relation

$$B\epsilon^{j\mu} = \frac{1 + j2c\dfrac{\omega}{\omega_n}}{-\omega^2/\omega_n^2} = \frac{1 + j2cd}{-d^2} \tag{9.39}$$

The output displacement is thus a sinusoidal function of time having an amplitude B and phase angle μ relative to the error, equal, respectively, to

$$\left\{\begin{aligned} B &= \frac{\sqrt{1 + 4c^2d^2}}{d^2} && (9.40) \\ \mu &= \tan^{-1} 2cd. && (9.41) \end{aligned}\right.$$

Locus of Output Displacement Vector.—As in the case of viscous output damping the output displacement vector varies in magnitude and phase, with respect to the error, when the frequency d of the error signal fed into the controller is varied (see Fig. 9.6). For any given fixed value of the damping ratio c the locus of the output vector can then be plotted, as was done in Fig. 9.10 for viscous output damped systems. In the case of purely error-rate-damped systems the locus assumes a parabolic form as shown on the right in Fig. 9.15. This is obtained by calculating the real and imaginary components of the output vector θ_o from Eq. (9.39) for various values of the frequency d, or by computing the magnitude B and phase angle μ of this vector from Eqs. (9.40) and (9.41).

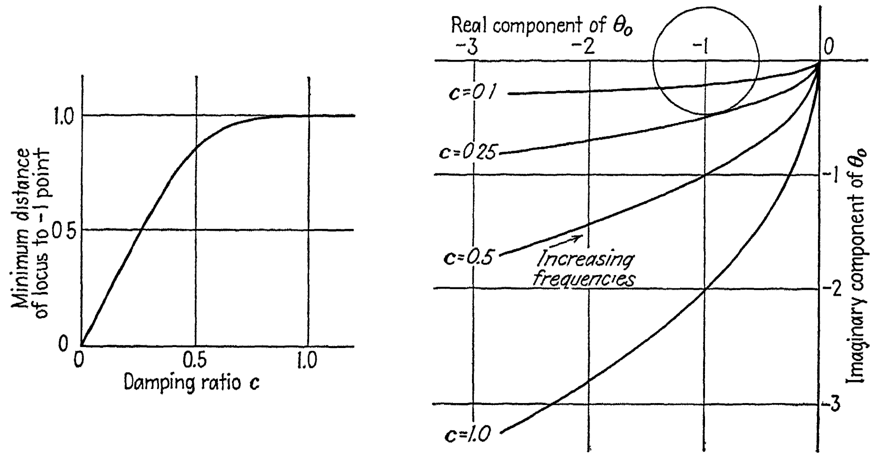

FIG. 9.15.—Output vector locus of error-rate-damped servomechanism.

It is seen that for low-frequency values the output vector has considerable magnitude and a phase angle approaching -180 deg. As the frequency is increased, the magnitude of the output vector decreases. However, the locus does not pass through the -1 point of the horizontal axis, and the system is stable. As in the case of viscous damping the smaller the value of the damping ratio c, the closer will the locus come to the horizontal axis, and, in particular, to the -1 point of this axis.

As before, the curve is entirely within the third quadrant of the diagram, and a simple way of judging the stability of the system is to measure the radius of the circle having the -1 point of the horizontal axis as its center and tangent to the locus of the output vector. This circle has been drawn in Figs. 9.10 and 9.15 for the curves corresponding to a damping ratio $c = 0.25$. The radius is approximately the same in both cases, which confirms the criterion postulated on page 221. This procedure is particularly helpful, as will be seen in the following chapter, for estimating the value of the damping ratio and degree

of stability of complicated systems, for which mathematical computations are sometimes complicated and lengthy. However, for such systems the *shape* of the curve must also be considered, as will be described later.

Amplitude-frequency Response of Output-error Function.—In order to introduce the frequency d explicitly in the graphs, the amplitude and phase of the output displacement must be computed and represented separately from Eqs. (9.40) and (9.41), respectively.

Referring first to the variation of the amplitude B with the frequency d, Eq. (9.40) shows that, for small values of d, the amplitude tends toward the value

$$B = \frac{1}{d^2}, \tag{9.42}$$

which can also be written

$$\log B = -2 \log d, \tag{9.43}$$

or

$$20 \log_{10} B = -12 \log_2 d, \tag{9.44}$$

from which

$$\frac{20 \log_{10} B}{\log_2 d} = -12 \text{ db per octave.} \tag{9.45}$$

Expressed in words, the curve representing the output displacement amplitude as a function of the frequency has, at low-frequency values, a negative slope of 12 db per octave.

Conversely, for large values of the frequency d Eq. (9.40) tends toward the value

$$B = \frac{\sqrt{4c^2d^2}}{d^2} = \frac{2c}{d}. \tag{9.46}$$

This can also be written

$$\log B = \log 2c - \log d \tag{9.47}$$

or

$$20 \log_{10} B = 6 \log_2 2c - 6 \log_2 d \tag{9.48}$$

Since for any given system the damping ratio c is fixed, the output amplitude, expressed in decibels in the left-hand member of Eq. (9.48), varies linearly with the relative frequency, expressed in octaves in the right-hand member of this same equation. In other words, the amplitude-frequency curve is a straight line. The slope of this line shows the variation rate of the output amplitude with the frequency, and is equal to

$$\frac{20 \log_{10} B}{\log_2 d} = -6 \text{ db per octave.} \tag{9.49}$$

These relations as expressed in Eqs. (9.45) and (9.49) are exactly the opposite of those which were found for the viscous output damped servomechanisms. They are represented by the graph of Fig. 9.16.

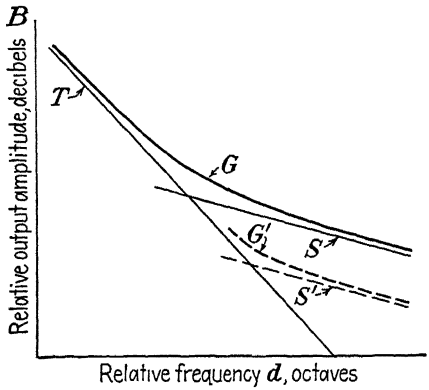

FIG. 9.16.—Output displacement amplitude (relative to error amplitude) as function of frequency in an error-rate-damped servomechanism.

It will be noted from Eqs. (9.42) and (9.46) that, at the crossing point of the two asymptotes T and S

$$\frac{1}{d^2} = \frac{2c}{d} \tag{9.50}$$

or

$$d = \frac{1}{2c}. \tag{9.51}$$

Also, contrary to the condition encountered with viscous output damping, if in an error-rate-damped system the damping ratio c is reduced to some smaller value c', the amplitude-frequency curve G and its 6 db per octave asymptote S are displaced downward, as shown in G' and S' in Fig. 9.16.

The amplitude-frequency relation of the servomechanism is determined experimentally in the same manner as for a servomechanism with viscous output damping, using the test setup illustrated in Fig. 9.6.

Phase-frequency Response of Output-error Function.—Variation of the phase angle of the output displacement with the error frequency is obtained from Eq. (9.41).

$$\mu = \tan^{-1} 2cd. \tag{9.41}$$

This relation shows at once that

$$\begin{array}{ll} \text{For } d \text{ small} & \mu \to -180 \text{ deg.} \\ \text{For } d = \dfrac{1}{2c} & \mu = -180 + 45 = -135 \text{ deg.} \\ \text{For } d \text{ large} & \mu \to -90 \text{ deg.} \end{array}$$

This is shown graphically in Fig. 9.17, which illustrates a variation opposite to that which was found for viscous output damped mechanisms (compare with Fig. 9.14). However, it will be noted that both in the viscous-damped and error-rate-damped servos phase shifts of -90 and -180 deg. correspond, respectively, to negative slopes of 6 and 12 db per octave for the amplitude-frequency curve. This relationship applies to all systems that, like the servomechanisms here considered, are comparable to feedback-connected amplifiers. The phase shifts mentioned,

which were shown to be indicative of the stability of the system, are the *minimum* values of phase shift obtaining in the system considered. They do not take into account additional phase shifts introduced by devices or networks that may be inserted in the servo loop such as transmission lines in which a phase shift may occur that is greater than that indicated by the attenuation introduced. Proper allowance for these additional phase shifts must be made when the stability of the system is being investigated. This will be illustrated in the following chapter.

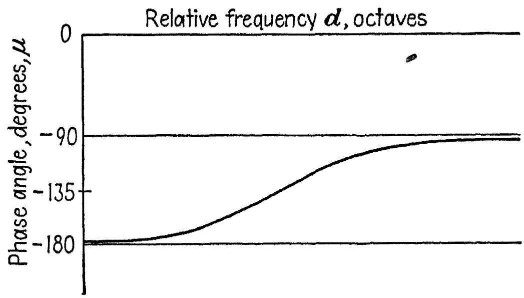

FIG. 9.17.—Output vector phase angle as function of frequency in an error-rate-damped servomechanism.

In the case of error-rate damping, as in the case of viscous output damping, the damping ratio c cannot be reduced below a value for which the phase angle exceeds -158 deg. when $d = 1$. This, in turn, corresponds to a position of the crossing point of the asymptotes about one octave *above* the natural frequency of the system. Here again, therefore, inspection of the frequency response curve immediately yields information on the stability of the system.

In error-rate-damped systems, the frequency at which the two asymptotes of the amplitude curve cross each other has a further significance, which relates this method of investigating the properties of the servo to the design factors discussed in Chaps. V and VII. Thus, for pure error-rate damping ($r = 0$) it was shown in Chap. VII that

$$\omega_b = \frac{\omega_n}{2c} \qquad \text{(7.17 for } r = 0\text{)}$$

where ω_b is the error frequency which determines the design of the error differentiating network. Substituting this in Eq. (9.39), the latter becomes

$$B\epsilon^{j\mu} = -\frac{\omega_n^2}{\omega^2} - j\frac{\omega/\omega_b}{\omega^2/\omega_n^2}. \qquad (9.52)$$

Just as in Chap. VII, it is then found that the real and imaginary terms of

this expression of the output vector are equal when the applied frequency is

$$\omega = \omega_b \tag{9.53}$$

and since

$$d = \frac{\omega}{\omega_n}$$

it follows that

$$\frac{\omega_b}{\omega_n} = d = \frac{1}{2c}. \tag{9.54}$$

Expressed in words, and by comparing with Eq. (9.51), the frequency ω_b for which the real and imaginary components of the output displacement vector are equal is the frequency which corresponds to the crossing point of the asymptotes of the amplitude-frequency curve of the output-error function.

SERVOMECHANISMS WITH COMBINED VISCOUS OUTPUT DAMPING AND ERROR-RATE DAMPING

Expression of Output Displacement as a Function of Input-output Error.—As for the viscous-damped and error-rate-damped servomechanisms, the output-error transfer function may be derived by either one of two methods. A sinusoidal error function of unit amplitude, as expressed in Eq. (9.19), and the corresponding output function, as expressed in Eq. (9.20), can be written in Eq. (6.1), which is the equation of motion of the system. Or else, the expression of the input-output function $A\epsilon^{j\lambda}$ given in Chap. VI as Eq. (6.52) can be written in Eq. (9.7a).

$$\left\{\begin{aligned} A\epsilon^{j\lambda} &= \frac{1 + j2c(1 - r)d}{1 - d^2 + j2cd} && (6.52) \\ B\epsilon^{j\mu} &= \frac{A\epsilon^{j\lambda}}{1 - A\epsilon^{j\lambda}}. && (9.7a) \end{aligned}\right.$$

The expression is then obtained

$$B\epsilon^{j\mu} = \frac{1 + 2jc(1 - r)d}{-d^2 + 2jrcd}. \tag{9.55}$$

This is the complex expression of the output displacement of the servomechanism, referred to the input-output error, when the latter is a sinusoidal function of time of unit amplitude. This output displacement has an amplitude B and phase angle μ equal, respectively, to

$$\left\{\begin{aligned} B &= \sqrt{\frac{1 + 4c^2(1 - r)^2d^2}{d^4 + 4r^2c^2d^2}}. && (9.56) \\ \mu &= \tan^{-1} 2c(1 - r)d + \tan^{-1} \frac{2rc}{d}. && (9.57) \end{aligned}\right.$$

It may be verified at once that for $r = 1$ and $r = 0$, these expressions reduce to those encountered before for servomechanisms having, respectively, pure viscous output damping and pure error-rate damping.

Locus of Output Displacement Vector.—As just pointed out, a servomechanism with combined viscous output and error-rate damping has some of the properties of mechanisms having only one kind of damping and also properties of mechanisms having the other kind of damping. The actual features depend upon the relative values of the factors c and r. They are not described here in greater detail, other than mentioning that, like the two classes of mechanisms studied in the preceding sections, the locus of the output vector extends in the third quadrant of the reference axes, without enclosing the -1 point of instability, under the positive damping conditions considered here.

Frequency Dependence of Amplitude and Phase of Output-error Function.—Reference being first made to Eq. (9.56), which expresses the output amplitude B as a function of the frequency d, it is seen that for small values of d the amplitude tends toward the value

$$B = \frac{1}{2rcd}. \tag{9.58}$$

For large values of d the amplitude tends toward the value

$$B = \frac{2c(1 - r)}{d}. \tag{9.59}$$

For intermediate values of d (in the vicinity of $d = 1$ and more markedly for small values of c) the amplitude tends to vary as

$$B = \frac{1}{d^2} \tag{9.60}$$

By comparing Eqs. (9.58) and (9.59), respectively, with Eqs. (9.30) and (9.46), it is seen that, except for the introduction of the factor r, the system has properties similar to those of a viscous-damped mechanism at the lower frequencies and similar to those of an error-rate-damped mechanism at the higher frequencies. In both cases, the amplitude varies inversely as the frequency, and this was found to correspond to a variation rate of 6 db per octave and to a phase shift of -90 deg.

On the other hand, for frequencies of intermediate values Eq. (9.60) shows that the amplitude varies inversely as the square of the frequency, which was seen to correspond to an amplitude variation rate of 12 db per octave. The phase shift then approaches -180 deg.

These limit values are represented graphically in Fig. 9.18 by the straight lines S_1, S_2, and T. The amplitude and phase angle curves G and H are also shown. The frequency corresponding to the crossing

point P_1 of the lines S_1 and T is obtained by equating the expressions, Eqs. (9.58) and (9.60).

$$\frac{1}{2rcd} = \frac{1}{d^2} \tag{9.61}$$

from which

$$d = 2rc. \tag{9.62}$$

Similarly, the frequency corresponding to the crossing point P_2 of the lines S_2 and T is found by equating the expressions, Eqs. (9.59) and (9.60).

$$\frac{2c(1-r)}{d} = \frac{1}{d^2} \tag{9.63}$$

from which

$$d = \frac{1}{2(1-r)c}. \tag{9.64}$$

The relations, Eqs. (9.62) and (9.64), should be compared to Eqs. (9.36) and (9.51). These equations (9.36) and (9.51) express the fre-

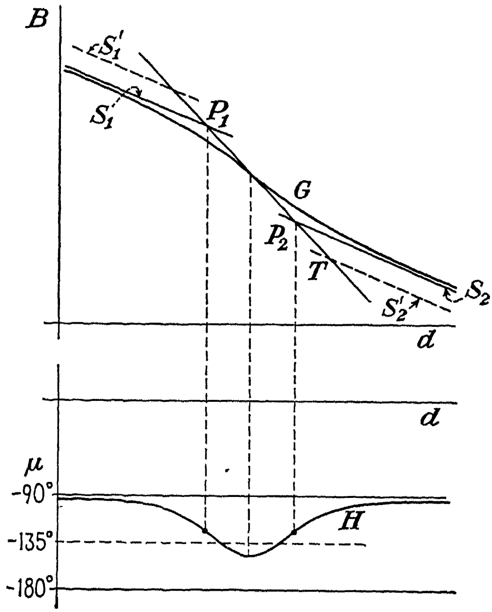

FIG. 9.18.—Output vector amplitude and phase angle as functions of frequency in a servomechanism with combined viscous damping and error-rate damping.

quency value d_p for which the phase angle μ is equal to -135 deg. in a servomechanism having *either* viscous damping only *or* error-rate damping only. It has been shown that, for what is here considered as the practical limit of acceptable stability, it was not permissible to reduce the damping ratio c below a value for which this frequency d_p would

differ by more than about one octave from the natural frequency of the system.

In the present instance of *combined* viscous damping and error-rate damping, if either one of the two frequency values expressed in Eqs. (9.62) and (9.64) is written in the phase angle equation (9.57), it is found at once that the angle μ corresponding to the points P_1, P_2 of Fig. 9.18 is comprised between -90 and -135 deg. At these frequencies, the system is therefore more stable than, at corresponding frequencies, a system with only one kind of damping. Expressed differently, in a system having both kinds of damping, it is permissible to use a smaller amount of each type of damping without impairing the stability of the mechanism. Lowering the value of c causes the 6-db sloping lines S_1 and S_2 to be displaced, respectively, upward and downward, as shown by the lines S'_1 and S'_2 in Fig. 9.18, and the points P_1 and P_2 to move farther apart. Thus, in the case where $r = \frac{1}{2}$, the frequencies corresponding to these points may differ by as much as two octaves from the natural frequency of the system, without exceeding the same practical stability limits as were obtained in a single-damped system with a frequency difference of only one octave.

SERVOMECHANISMS WITH INTEGRAL CONTROL

A discussion of servomechanisms with integral control can be made along lines similar to those followed previously for the other types of systems considered. This will be sketched only briefly here, since a complete study would be beyond the scope of the present chapter in view of the many parameters involved. Since the output-input function $A\epsilon^{j\lambda}$ was not derived previously, the output-error transfer function will be obtained directly from the original differential equation of the system.

The equation of motion of a servomechanism with integral control was shown in Chap. VIII to be

$$K\theta + L\frac{d\theta}{dt} + N\int \theta\,dt = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt}. \tag{8.1}$$

If the error θ is a sinusoidal function of time of unit amplitude

$$\theta = \epsilon^{j\omega t}, \tag{9.65}$$

the output displacement, as before, will be

$$\theta_o = B\epsilon^{j(\omega t+\mu)}. \tag{9.66}$$

Writing these expressions and their respective time derivatives and integral in Eq. (8.1), this becomes

$$K\epsilon^{j\omega t} + j\omega L\epsilon^{j\omega t} - j\frac{N}{\omega}\epsilon^{j\omega t} = -\omega^2 JB\epsilon^{j(\omega t+\mu)} + j\omega FB\epsilon^{j(\omega t+\mu)}. \tag{9.67}$$

Dividing through by $K\epsilon^{j\omega t}$ and collecting terms

$$B\epsilon^{j\mu} = \frac{1 + j[\omega(L/K) - (N/K\omega)]}{-\omega^2(J/K) + j\omega(F/K)}. \tag{9.68}$$

Introducing the factors c, d, r, and s previously defined, this equation is finally written

$$B\epsilon^{j\mu} = \frac{1 + j\left[2c(1 - r)d - \dfrac{s}{d}\right]}{-d^2 + 2jrcd}. \tag{9.69}$$

This is the complex expression of the output displacement when a sinusoidal error function of unit amplitude is fed into the controller. The output displacement thus has an amplitude B and phase angle μ, relative to the error, equal, respectively, to

$$B = \sqrt{\frac{1 + \left[2c(1 - r)d - \dfrac{s}{d}\right]^2}{d^4 + 4r^2c^2d^2}} \tag{9.70}$$

$$\mu = \tan^{-1}\left[2c(1 - r)d - \frac{s}{d}\right] + \tan^{-1}\frac{2rc}{d}. \tag{9.71}$$

The last three expressions differ only by the addition of the integral control term $-s/d$ from the corresponding similar Eqs. (9.56, 9.57, and 9.58) obtained for a servomechanism with combined viscous output damping and error-rate damping. This additional term $-s/d$ becomes negligibly small for the larger values of the frequency d. At these higher frequency values, therefore, the system has substantially the same operating properties as in the absence of integral control. In other words, the integral control is particularly effective at lower operating frequencies.

For small values of the frequency Eqs. (9.70) and (9.71) then tend toward the values

$$B = \frac{s}{2rcd^2} \tag{9.72}$$

$$\mu = \tan^{-1}\left(-\frac{s}{d}\right) + \tan^{-1}\frac{2rc}{d}. \tag{9.73}$$

Equation (9.72) shows that at these lower frequencies the output amplitude varies inversely as the square of the frequency, at a rate of 12 db per octave. On the other hand, the phase angle, as expressed in Eq. (9.73), approaches -180 deg. These conditions, however, do not make the system unstable if the natural frequency is sufficiently higher than the low frequencies at which the integral control is appreciably effective.

The graphical representation is illustrated in Fig. 9.19, which may be readily understood by comparison with the graphs of Fig. 9.18.

As in the case of error-rate damping, where the point P_2 was found to correspond to the frequency ω_b, the point P_3 of the integral control

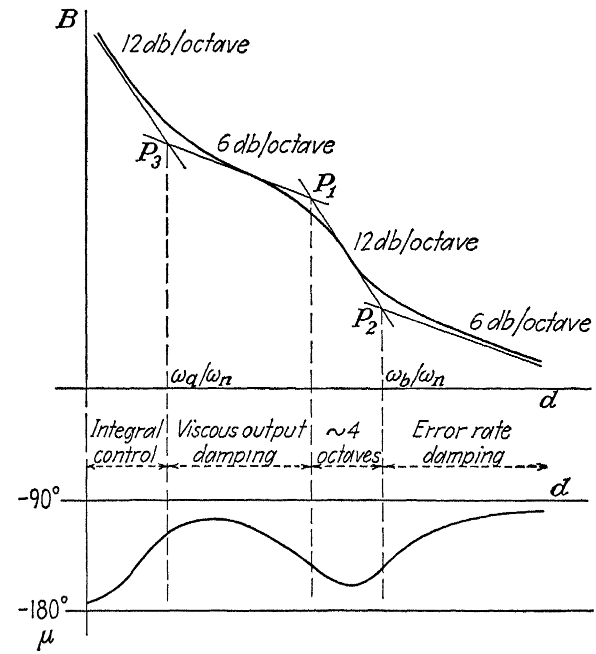

FIG. 9.19.—Output vector amplitude and phase angle as functions of frequency in a servomechanism with viscous damping, error-rate damping, and integral control.

asymptote can readily be shown to correspond to the frequency ω_q defined in Chap. VIII as determining the design of the integral control circuit.

OPERATING LIMITATIONS OF SERVOMECHANISMS AS SHOWN BY TRANSFER FUNCTION CHARACTERISTICS

Consider a servomechanism of the general pattern shown in Figs. 9.5 and 9.6, and suppose that there is neither viscous friction nor error-rate damping. In particular, suppose that the controller output torque is directly proportional to the error voltage and is independent of the output speed. The driving torque $K\theta$ is then equal to the inertia retarding torque $J(d^2\theta_o/dt^2)$, which is the only retarding torque operating in the system. The equation of motion is then

$$K\theta = J\frac{d^2\theta_o}{dt^2}. \tag{9.74}$$

For a sinusoidal error of unit amplitude

$$\theta = \epsilon^{j\omega t}, \tag{9.75}$$

the output displacement, as before, is expressed

$$\theta_o = B\epsilon^{j(\omega t+\mu)}, \tag{9.76}$$

and Eq. (9.74) then becomes

$$K\epsilon^{j\omega t} = -\omega^2 JB\epsilon^{j(\omega t+\mu)}. \tag{9.77}$$

Dividing through by $K\epsilon^{j\omega t}$:

$$1 = -\omega^2 \frac{J}{K} B\epsilon^{j\mu} \tag{9.78}$$

or

$$1 = -\frac{\omega^2}{\omega_n{}^2} B\epsilon^{j\mu} = -d^2 B\epsilon^{j\mu} \tag{9.79}$$

from which

$$B\epsilon^{j\mu} = -\frac{1}{d^2}. \tag{9.80}$$

From this expression it is seen that the output displacement has an amplitude B and phase angle μ equal, respectively, to

$$B = \frac{1}{d^2} \tag{9.81}$$

$$\mu = -180 \text{ deg.} \tag{9.82}$$

Equation (9.81) can be written

$$20 \log_{10} B = -12 \log_2 d \tag{9.83}$$

showing, as before, that the output amplitude decreases by 12 db when the frequency is increased by 1 octave.

This is represented graphically by the straight line T in Fig. 9.20. In view of the -180 deg. phase shift of the output displacement with respect to the error, the system is therefore unstable when the servo loop is closed, as in Fig. 9.5. Thus, the system oscillates continuously at its natural frequency ($B = 0$ db, $d = 0$ octave) in the absence of any input displacement.

In order to stabilize the system, viscous output damping may be introduced. This depresses the characteristic line T and reduces its slope from -12 to -6 db per octave, as shown by the line S_1. The phase angle is then reduced to -90 deg. If this is the only stabilization means employed, the crossing point P_1 of the two lines T and S_1 must correspond to a frequency no smaller than one octave below the natural frequency ($d = 1$) of the system.

If error-rate damping is used concurrently with viscous output damping, the latter may be reduced to a point at which the frequency corresponding to P_1 will be as much as two octaves or more below the natural frequency of the system. This has the advantage of reducing the steady-state error.

The effect of error-rate damping is to raise the characteristic line T and reduce its slope from -12 to -6 db per octave, as shown by the line S_2 in Fig. 9.20 and to reduce the phase angle to -90 deg. The frequency

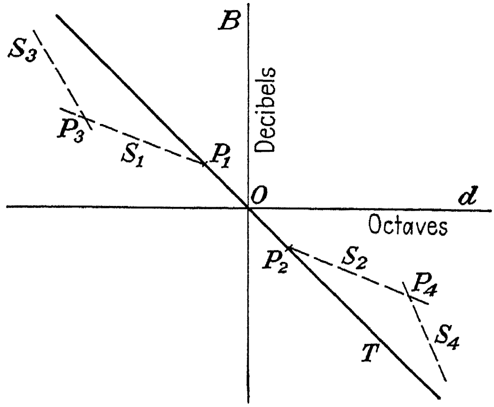

FIG. 9.20.—Effect of various types of damping and limiting on slope of transfer-function characteristic.

corresponding to the point P_2 should not be more than one or two octaves above the natural frequency, depending, respectively, on whether error-rate damping is used alone or in combination with viscous output damping.

If the servomechanism is intended to drive a substantial output friction load, which may perhaps vary at times, more gain (*i.e.*, more torque) may be required at the lower frequencies. This is obtained by adding integral control to the system, which raises the output amplitude and increases the slope of the characteristic, as shown by the line S_3 in Fig. 9.20. The approximate slope in this region is then 12 db per octave, but may be higher than this. The phase angle is -180 deg. or more, and the system may lose some stability if the frequency at the point P_3 is not sufficiently far below the natural frequency of the system.

Finally, another departure from the ideal characteristic line T occurs when the operating frequency (error frequency in the arrangement of Fig. 9.6, or input frequency in that of Fig. 9.5) approaches the carrier frequency (synchro-excitation frequency). The reason is that the servo motor, under such conditions, will fail to operate properly, its torque falling off abruptly as the frequency is increased. This is shown by the line S_4, which for this reason may have a very great slope.

These limiting factors at high operating frequencies are the inductance of the motor and transformer windings, which also reduces the torque and increases the slope of the characteristic curve in the region of the line S_4. A similar limitation may be due to the amplifier characteristics (the response of certain types of dynamo-electric amplifiers is limited to a maximum rate of some 2,000 volts per sec.).

Finally, the performance of the system may be reduced by the fact that only a limited torque can be obtained at low frequency from the integrating network, or at high frequency from the error differentiating network.

All these discrepancies will tend to alter the shape of the output characteristic, and this may therefore serve as a simple and accurate indication of the properties and performance of the servomechanism as a whole and of its individual components.

STABILITY CRITERIA OF OUTPUT VECTOR LOCUS DIAGRAM

For the simple viscous output damped and error-rate-damped servomechanisms with positive damping ratio considered previously, it was found that the locus of the output vector (referred to the unit error vector) extends entirely in the third, or lower left-hand, quadrant of the diagram. Such systems are inherently stable, by which is meant that a transient initiated by some variation of the input speed will eventually be damped out. However, the degree of stability of the system depends on the effective damping ratio; the amplitude and the duration of the transient vary in direction opposite to this ratio, as was explained in detail in the preceding chapters. It was also shown, in relation to Figs. 9.10 and 9.15, that the output vector locus will come closer to the -1 point of the horizontal axis, as the oscillation tendency of the system increases. This feature of the output vector locus will be illustrated presently, before considering the stability criteria of systems leading to more complicated types of locus curves.

Consider a simple servomechanism with pure viscous output damping having a damping ratio $c = 0.25$. Its output-error transfer function is expressed by Eq. (9.27) and represented by the curve A of Fig. 9.21, which is a reproduction of the curve $c = 0.25$ of Fig. 9.10.

Suppose now that integral control is added to the system, characterized by an integral constant $s = 0.29$, corresponding to an effective damping ratio $g = 0.10$. The new transfer function of the system is then expressed by Eq. (9.69), in which $c = 0.25$, $s = 0.29$, and $r = 1$, since there is no error-rate damping. Plotting the output vector locus from this equation, curve B of Fig. 9.21 is obtained. As called for by the new damping ratio $g = 0.10$, this curve runs substantially closer than curve A to the -1 point of the horizontal axis. This, in turn, denotes a cor-

responding reduction of stability of the system, in which the transient will die out less rapidly than in the original system represented by the curve A.

In order to restore the original stability, sufficient error-rate damping will now be added to the system, to raise the damping ratio to a value $g = 0.25$. Keeping the output friction the same, this is achieved by using, in Eq. (9.69), the values

$$c = 0.4 \qquad r = 0.625 \qquad \text{(and } s = 0.29 \text{ as before).}$$

The corresponding plot of the output vector locus is shown by the curve C of Fig. 9.21. Although different from the original curve A, this curve

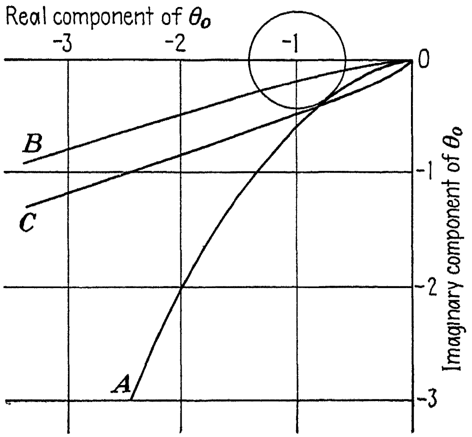

FIG. 9.21.—Stability criterion of output vector locus diagram.

C passes approximately at the same distance from the -1 point as this curve A, both curves being tangent to approximately the same circle drawn with the -1 point as its center. This further illustrates the fact that systems for which the output vector locus passes at a same distance from the -1 point will have the same degree of stability, and that the rate of decay of transients will be the same in these systems.

Output Vector Locus in Other Quadrants than the Third.—It has been shown that the shape, slope, and orientation of the output vector locus depends on the particular combination of the various kinds of damping and controls incorporated in the servomechanisms considered. As will be seen in the following chapter, other combinations and the introduction of additional factors may further alter the shape of the output vector locus. Contrary to the cases encountered heretofore, the locus may even extend into areas of the diagram located outside the third quadrant.

While the radius of the circle which has the -1 point of the horizontal axis as its center and which is tangent to the locus is then still an indica-

tion of the damping ratio of the system, this damping ratio may then be positive or negative, and the system is correspondingly stable or unstable, depending on the particular shape and disposition of the curve.

A rule permitting one to ascertain the stability or unstability of simple systems is illustrated below.[1] Let the curve A, Fig. 9.22, represent the output vector locus of the system considered. Let M be a line starting at the point of origin of the coordinate axes and extending down along

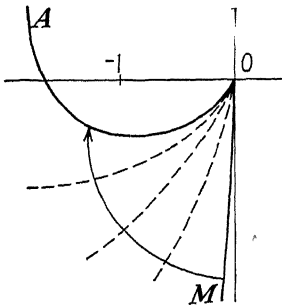

Fig. 9.22.—Output vector locus of stable servomechanism.

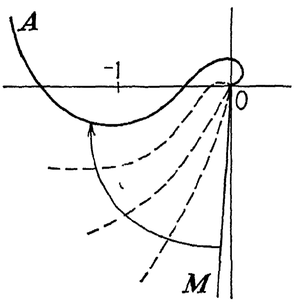

Fig. 9.23.—Output vector locus of stable servomechanism.

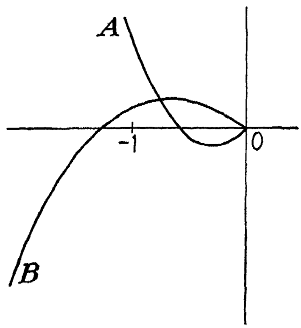

Fig. 9.24.—Output vector loci of unstable servomechanisms.

the negative portion of the imaginary (*i.e.*, vertical) axis into the third quadrant of the diagram. The system will be stable if as is the case here the line M can be fitted onto the curve A through clockwise rotation and suitable distortion, *without* sweeping over the -1 point of the horizontal axis.

According to this same rule the curve A of Fig. 9.23 corresponds to a stable system, while the curves A and B of Fig. 9.24 correspond to unstable systems.

[1] A great number of systems encountered in practice will fall into this class. Systems whose transfer function amplitude increases, at very low frequencies, at a rate greater than the inverse fourth power of the frequency may fail to follow this rule.

CHAPTER X

TYPICAL DESIGN CALCULATIONS AND GENERAL CONSIDERATIONS

The discussion in the preceding chapters relates essentially to a study of servomechanisms, which are control systems employed for the positioning of remote objects. Other control systems will be mentioned in the latter part of this chapter.[1] For the several types of servo systems studied, relations have been established between the fundamental parameters that characterize and identify their operation. From these relations and the curve diagrams that accompany them it is possible to design a servomechanism to meet specified requirements.

In this connection it will be noted that no specific design has been developed for the individual elements that make up the system, such as the controller amplifier, servo motor, gears, etc. Such particular design problems do not relate specifically to servo system design, but properly belong to specialized branches of engineering, which do not come within the scope of this book. The discussion was purposely limited to the definition of the operating requirements of these elements in order that once assembled into a servomechanism the desired system performance may be obtained.

Thus, for example, it was shown how to calculate the required amplifier gain and frequency bandwidth. But the actual design of the amplifier itself was not developed further, since many different types of amplifiers having the prescribed gain and frequency bandwidth will answer the purpose, providing that the amplifier will supply sufficient power to actuate the servo motor. Depending on the particular conditions of the system, a d-c or an a-c vacuum tube amplifier may be used, or some form of hydraulic on electrodynamic amplifier. Then again, the controller gain having been calculated in accordance with the methods outlined in this book, a mechanical torque amplifier may be used at the output shaft of the motor.[2]

[1] Applications of allied control engineering fields are discussed in the following books: SMITH, ED SINCLAIR, "Automatic Control Engineering," McGraw-Hill Book Company, Inc., New York, 1944; ECKMAN, D. P., "Principles of Industrial Process Control," John Wiley & Sons, Inc., New York, 1945.

[2] A comparative study of the relative advantages, in a particular case, of a motor input amplifier and output torque amplifier may be found in H. L. Hazen, Design and Test of a High-performance Servomechanism, *J. Franklin Inst.*, vol. 218, pp. 543–580, November, 1934.

Similarly, determination of the characteristics of the servo motor (torque, speed, moment of inertia, etc.) is explained in the text, but not the methods of designing the motor. This may be a d-c or an a-c motor, depending again on the particular circumstances of the system considered. A study of the motor design, like that of the amplifier, would unduly increase the scope of this book and would probably cover many topics available elsewhere in the published specialized literature.

Moreover, the detail development of particular cases, numerous as these may be, would nevertheless form an incomplete picture of servo design methods. However, in order to show some aspects of the procedure followed, a more complete example of design calculation of a particular servo system is given below.

Detailed Calculation of an Error-rate-damped Servo System with Static Friction Load.—The problem to be treated here relates to the design of a servo control system intended, by means of a handcrank, to rotate an asymmetrical radio antenna structure at a speed of 5 r.p.m. The antenna has a moment of inertia of 1,600 lb.-ft.^{2}, and may be subjected to wind, which will produce a maximum load of 500 ft.-lb. on the output shaft. The input-output position error due to loading must not exceed an arc of 10 min. at full load.

A system with combined viscous output damping and error-rate stabilization will be used, employing a synchro-repeater follow-up link and a 60-cycle, two-phase induction motor. Viscous output damping is to be provided by the negative slope of the torque-speed motor characteristic. Error-rate damping is obtained through a parallel-T filter network. The general layout of the system is shown in Fig. 10.1. It is similar to that discussed in Chaps. VI and VII, except that, in view of the small error angle specified, greater accuracy is obtained by coupling the output shaft to the follow-up link through a 36:1 step-up gear train. The error voltage is thus developed at a rate of 36 volts per deg. error. The supply line energizing the system is a single-phase, 60-cycle, 115-volt line.

1. *Motor Size.*—It is assumed that the servo motor will be coupled to the output load through a gear train having an efficiency of 80 per cent. Applying the formula

$$\text{Motor horsepower} = \text{speed in r.p.m.} \times \text{load in in.-oz.} \times 10^{-6}$$

the required motor rating is given by

$$0.8 \times \text{hp.} = 5 \times 500 \times 16 \times 12 \times 10^{-6}$$

from which

$$\text{Motor horsepower} = 1.25 \times 500 \times 192 \times 5 \times 10^{-6} = 0.6 \text{ hp.}$$

This is equivalent to

$$0.6 \times 746 = 450 \text{ electrical watts.}$$

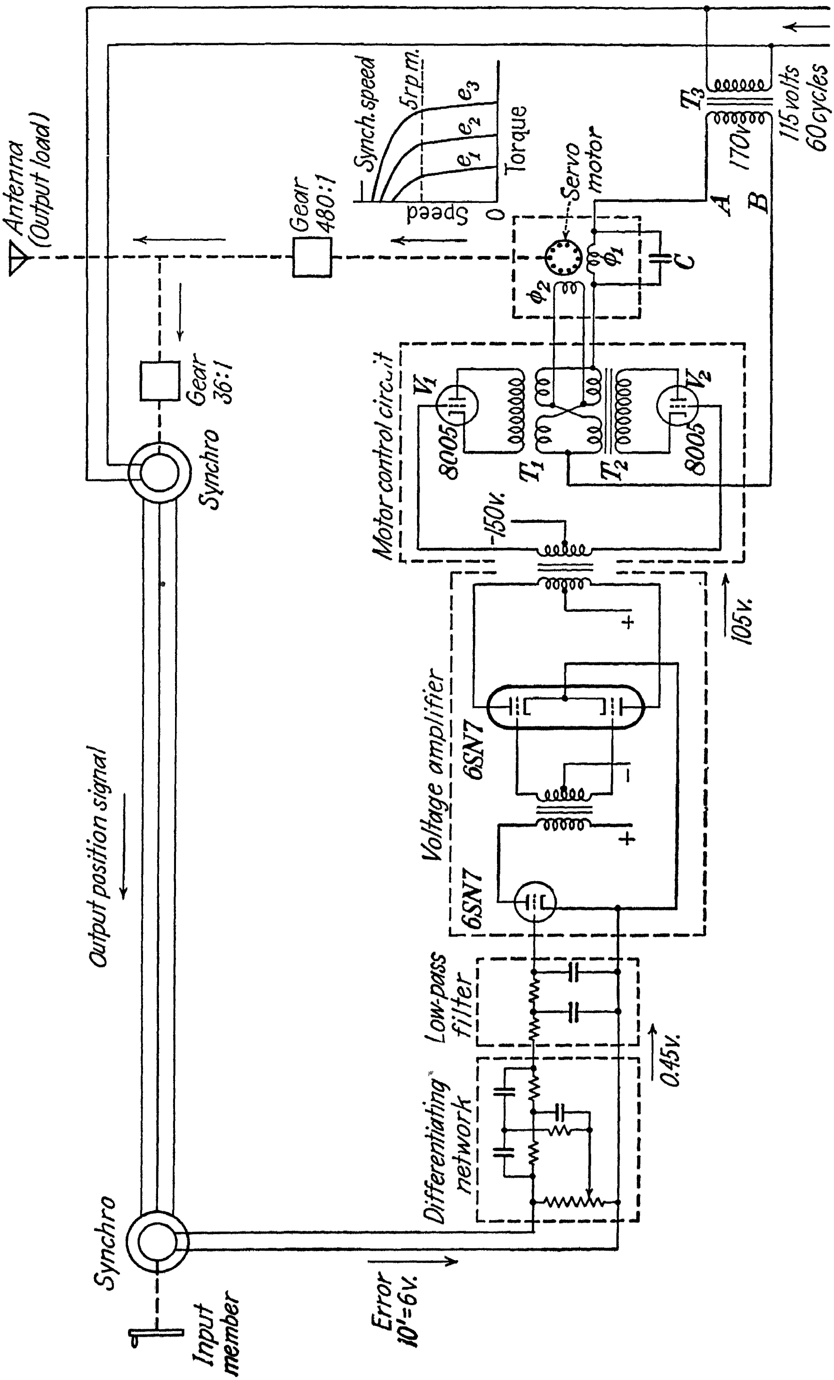

FIG. 10.1.—Simplified schematic diagram of a position-control servomechanism with viscous output damping and error-rate damping.

Among the motors available on the market,[1] one will be chosen that has the following approximate characteristics

Motor frame No. . .	143
Nominal rating. . .	½ hp.
Moment of inertia. . . .	270×10^{-6} slug-ft.²
Locked torque (115 volts)	1.97 ft.-lb
Torque/inertia ratio.	7,450

The motor is a two-phase, 60-cycle induction motor delivering 63 per cent of locked torque at 55 per cent of synchronous speed (3,600 r.p.m.), when operated on a two-phase supply.

2. *Gear Ratio.*—Taking 2,400 r.p.m. as a maximum servo operating speed for the motor, an output speed of 5 r.p.m. will require a motor output gear of ratio

$$N_m = \frac{2{,}400}{5} = 480.$$

3. *Controller Torque.*—The present system is subjected to an error due to external load. The torque output of the motor will be used to overcome the maximum output load torque of 500 ft.-lb. on the antenna.

$$\text{motor torque} = \text{torque due to wind.}$$

Since the error of 10 min. [or $^{10}/_{60}$ deg., or $10/(60 \times 57.3)$ radian] permitted under these conditions corresponds to the load of 500 ft.-lb., the controller factor K is calculated from the relation

$$K = \frac{\text{torque}}{\text{error}} = \frac{500}{\left(\dfrac{10}{60 \times 57.3}\right)} = 171 \times 10^3 \text{ ft.-lb. per radian error angle.}$$

4. *Moment of Inertia and Natural Frequency.*—The moment of inertia of the load, expressed in slug-ft.², is equal to

$$J_L = \frac{1{,}600}{32} = 50 \text{ slug-ft.}^2$$

The moment of inertia of the motor is given as

$$J_M = 270 \times 10^{-6} \text{ slug-ft.}^2,$$

which, referred to the output shaft, is equal to

$$J_{MO} = 270 \times 10^{-6} \times 480^2 = 62 \text{ slug-ft.}^2$$

[1] Diehl Manufacturing Company, Finderne, N.J.

The total moment of inertia of the system, referred to the output shaft, is then[1]

$$J = 50 + 62 = 112 \text{ slug-ft}^2.$$

The natural frequency of the system is then

$$\omega_n = \sqrt{\frac{K}{J}} = \sqrt{\frac{171 \times 10^3}{112}} = 39 \text{ radians per sec.}$$
$$= 6.25 \text{ cycles per sec.}$$

5. *Error Signal Frequency Band.*—A damping ratio $c = 0.35$ is chosen arbitrarily. It will be pointed out in the next paragraph entitled *Motor Control Network* that the motor is used in a bridge-type control circuit with finite impedances. Under these conditions, the torque-speed characteristic curves differ over the useful range from those obtained when the motor is energized from a supply source with zero impedance. The small graph at the right of Fig. 10.1 shows a set of typical torque-speed curves for a motor fed from the bridge circuit considered. From these curves it appears that the torque decreases by about 10 per cent when the output speed increases from zero to 5 r.p.m., corresponding to a motor speed of about 2,400 r.p.m. at the operating torque. The effective friction introduced by the motor characteristic is then equal to

$$F = \frac{\text{torque variation (in ft.-lb.)}}{\text{speed variation (in radians per sec.)}} = \frac{500 \times 0.1}{5 \times \frac{2\pi}{60}} = 100 \text{ (approximately).}$$

Applying Eq. (7.18) of Chap. VII, the relative friction r is then equal to

$$r = \frac{F}{2c\omega_n J} = \frac{100}{2 \times 0.35 \times 39 \times 112} = 0.03.$$

Thus, the relative friction constant may, if desired, be considered as negligibly small. The error-rate network parameter is then

$$\omega_b = \frac{\omega_n}{2c(1 - r)} = \frac{6.25}{2 \times 0.35 \times (1 - 0.03)} = 9 \text{ cycles,}$$

and the relative frequency bandwidth is

$$x_b = \frac{2\omega_b}{u_c} = \frac{2 \times 9}{60} = 0.3.$$

[1] It is seen that the moments of inertia of the load and motor are of the same order of magnitude, when referred to a common shaft. It is then permissible to calculate on the basis of load torque alone. If the inertia of the load were substantially greater than that of the motor, a larger motor might be required, even if the normal external load torque does not necessitate this, in order that the motor may supply the necessary torque for maximum acceleration under conditions of sudden input speed variations.

6. *Motor Control Network.*—As was explained in Chap. VII, the output of the stabilizing network (notch filter) is a 60-cycle voltage, the amplitude of which is proportional to the sum of the error and error rate of change. This voltage is amplified, and then applied to the motor control circuit. The output torque of the motor must be proportional to the voltage fed to the controller, and must reverse direction when the voltage reverses phase.

Fundamentally, the principle involved consists in controlling the induction motor by means of variable impedances in series with the motor

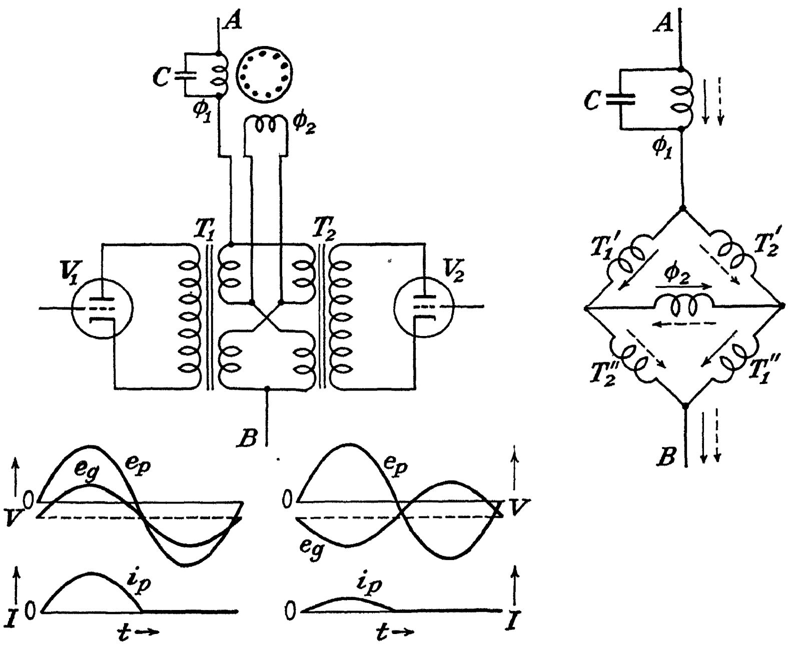

FIG. 10.2.—Motor control network.

windings. One phase is fed from the output of a bridge circuit that may be unbalanced in either direction, depending on the voltages applied to two control tubes. The other phase is connected in series with the p wer circuit feeding the bridge. The basic circuit arrangement is illustrated on the left-hand side of Fig. 10.2. A simplified schematic of the circuit is shown on the right-hand side of the figure.

The circuit consists of the two-phase induction motor, a pair of power control transformers, a pair of control tubes, and a phasing capacitor connected as shown. In the upper part of the left-hand portion of Fig. 10.2 the motor is represented by the circular rotor and two stator coils ϕ_1 and ϕ_2, oriented at right angles; these constitute the two phases of the motor. The control transformers are wound with double pri-

maries, and have a high step-up ratio to the secondary. In this manner, the low-voltage, high-current motor may be matched to high-voltage, low-current control tubes. The transformer primaries are cross-connected, and the center points are tied to phase 2 of the motor. The entire combination is connected to the 60-cycle supply line at points A and B.

The three-electrode vacuum tubes V_1 and V_2 are connected across the secondaries of the power control transformers. When these tubes are not conducting, they reflect a high impedance into the primaries of the control transformers. When they are conducting, the effective impedance transferred to the primary windings is reduced according to the amount of current passed by the tubes. The impedance shown in the primaries of each of the transformers may be varied independently by controlling the grid voltages of the control tubes.

The effect of varying the impedance of the transformers may be explained with reference to the simplified schematic diagram located at the right-hand side of Fig. 10.2. This diagram shows the low-voltage connections of the left-hand diagram, rearranged in a form that makes the operation more readily understandable. Power is again supplied across the entire circuit, *i.e.*, between points A and B. The two primary windings of the transformer T_1 are designated as T'_1 and T''_1, respectively, while the primaries of the transformer T_2 are labeled T'_2 and T''_2. The phasing capacitor C is connected across the motor winding ϕ_1 so that the current in this winding is approximately 90 deg. out of phase with the current in the other motor winding ϕ_2, as required for the motor to operate. The winding ϕ_2 is connected to the mid-points of the bridge circuit, the primaries of each separate transformer being on opposite sides of the bridge.

When both transformers have the same impedance, a limited current flows through the motor winding ϕ_1, but since the bridge is balanced, no current flows through the winding ϕ_2. As the torque produced by the motor is proportional to the *product* of the currents in the two windings, no torque will be produced by the motor in this balanced condition.

Suppose now that the impedances of the primary windings of the transformer T_1 are lowered and the impedances of the primaries of the transformer T_2 are raised. If the instantaneous a-c flow is considered to be in the direction from A to B, then the current will flow downward through the winding ϕ_1. Upon reaching the bridge, the current will tend to flow in the direction indicated by the solid arrows, through the first primary T'_1 of the transformer T_1, to the right through the motor winding ϕ_2, then downward through the second primary T''_1 of the transformer T_1. As the primaries of the transformer T_2 have a higher imped-

ance than those of the transformer T_1, there will be less tendency for current to flow through these coils. A quadrature phase relationship between the two motor windings being maintained by the capacitor C, the motor produces a torque proportional to the product of the currents in the two motor windings. This, in turn, is a function of the degree of unbalance of the bridge circuit.

Conversely, under the condition in which the primaries of the transformer T_2 have a lower impedance than those of the transformer T_1, the current will still tend to flow downward through the winding ϕ_1 of the motor. However, upon reaching the bridge circuit, the current will tend to flow in the direction of the dotted arrows, downward through the first primary T'_2 of the transformer T_2, to the left through the motor winding ϕ_2, and then downward through the second primary T''_2 of the transformer T_2. There is less tendency for the current to flow through the primaries of the transformer T_1. Again, the motor torque is proportional to the product of the currents flowing through the two motor windings. However, the current flowing through the motor winding ϕ_2, now assuming a reversed direction, produces a reversal of the torque developed by the motor.

Thus, a means is provided for controlling the torque produced by an induction motor, since the magnitude of the torque is proportional to the amount of unbalance of the bridge circuit, and the direction of the torque is a function of the direction of unbalance. The technique used to vary the impedances that the tubes reflect into the primaries of the control transformers is as follows:

The alternating voltages that appear across the primaries of the transformers are stepped up and applied to the plates of the tubes. From the symmetry of the circuit, it is seen that the tubes are thus fed in phase, *i.e.*, in a push-push relation. In Fig. 10.2, this plate voltage e_p is depicted in the small graphs directly below each tube. The current will flow during those half cycles when the plates of the tubes are positive, in amounts which depend on the instantaneous grid voltages of the tubes.

On the other hand, the grids of the tubes are fed in a push-pull relationship. The grid voltage e_g is also shown on the graphs. It will be noted that in the left-hand tube the grid voltage is in phase with the plate voltage, while in the right-hand tube the grid voltage is in phase opposition with the plate voltage. Accordingly, current will flow through the tube V_1 while no current, or only a negligibly small current, will flow through the tube V_2.

It follows that the impedance reflected into the primaries of the transformer T_1 will be lowered, while that reflected into the primaries of the transformer T_2 will be raised, as compared to the equal impedances that exist when no alternating voltage is applied to the grids. If the grid

voltage on each tube is reversed in phase, the primary impedances of the transformer T_1 will be increased, and those of the transformer T_2 will be decreased. The amount of unbalance, and therefore the magnitude of the motor torque, is dependent on the amplitude of the grid voltage. Hence, this motor control circuit performs the function of producing motor torque, controlled in amount and direction by the amplitude and phase of the 60-cycle control voltage applied to the grids of the tubes.

The control voltage is obtained from the amplifier, as shown in Fig. 10.1. The direction of the torque applied to the load is always such as to reduce the error between the antenna and input handcrank positions and to damp out such oscillations as may occur.

Referring now to the quantitative design of the network, it was found, in Sec. 1 above, that the motor output is 450 watts. Assuming that the motor efficiency is of the order of 50 per cent and the motor power factor approximately equal to 0.6, the motor would consume 900 watts and 1,500 volt-amp.

With a voltage of 115 volts applied to the motor windings,[1] this corresponds to a current of 750/115 = 6.4 amp. flowing through each motor winding. Actually, under maximum unbalance conditions of the transformer bridge network, no more than 100 volts appear across the bridge terminals (motor winding ϕ_2) with 120 and 20 volts across the windings T'_1 and T'_2, respectively.

For the control tubes V_1 and V_2, it is convenient to use RCA type 8005 triodes. As shown by the curves of Fig. 10.2, these tubes are used under conditions similar to those of *class C amplifier* operation, for which the manufacturer's tube data indicate a maximum permissible plate voltage of 5,700 volts. This peak voltage corresponds to 4,000 volts r.m.s. With 120 volts applied to the primary winding of the transformer, as mentioned in the preceding paragraph, this implies a transformer turns ratio of approximately 34:1.

FIG. 10.3.—Waveshape of plate current in control tubes.

The primary current having been calculated as 6.4 amp., the secondary current is then equal to 6.4/34 = 190 ma. This current has substantially the wave shape shown in Fig. 10.3, composed of a succession of half-sinusoids. Fourier analysis of this wave shape allows the current to be expressed

$$i = I_{\max}\left(\frac{1}{\pi} + \frac{1}{2}\cos\omega t + \frac{2}{3\pi}\cos 2\omega t \cdots\right).$$

[1] Since the motor windings are in series with the impedance of the control transformers T_1 and T_2, the voltage loss across this impedance must be compensated for by connecting the circuit to the supply line through a transformer T_3, which steps the supply voltage up to some 170 volts. The rated voltage of 115 volts is thereby obtainable at the terminals of the motor windings.

The 60-cycle component of the current is thus equal to

$$I_{\text{r.m.s.}}\ (60\text{-cycle}) = \frac{I_{\max}}{2} \times \frac{1}{\sqrt{2}}.$$

On the other hand, under maximum unbalance conditions, the primary voltage of the transformer connected to the tube having the least negative grid voltage becomes as small as 20 volts. This corresponds to a secondary voltage of 680 volts r.m.s., or 950 volts peak value. From the vacuum tube characteristic curves it is found that this peak voltage,

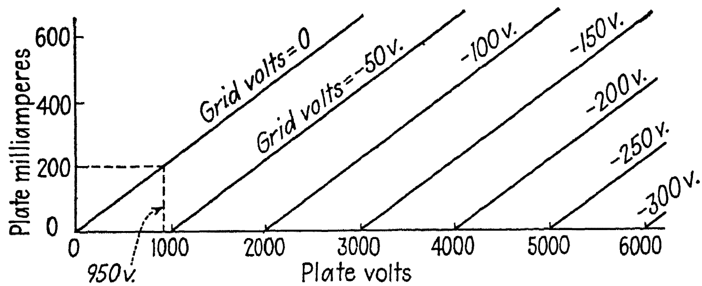

FIG. 10.4.—Approximate characteristics of control tubes.

in turn, corresponds to a peak plate current of 200 ma., when the grid voltage is equal to zero. The r.m.s. value of the 60-cycle plate current of the tube is then

$$I = \frac{200}{2\sqrt{2}} = 70 \text{ ma.}$$

Since a secondary current of 190 ma. is required, it is then necessary to connect three tubes in parallel, in place of each one of the tubes V_1 and V_2 shown in Fig. 10.2.

From the tube characteristic curves it is also seen that in order to unbalance the transformers in the opposite direction by reducing to zero the tube plate current for a peak plate voltage of 5,700 volts the grid of the tube must be given a negative voltage of approximately -300 volts. Since the grid voltage swing ranges over approximately ± 150 volts, a bias voltage of -150 volts must be applied in order to keep the grid negative at all times. The r.m.s. grid voltage is equal to $150/\sqrt{2} = 105$ volts.

7. *Amplifier Gain.*—The maximum permissible input-output position error of 10 min. develops full controller torque, and thus must produce maximum unbalance of the control tubes. In other words, this error must cause a voltage of 105 volts r.m.s. to appears at the amplifier output terminals, to be applied to the grids of the power control tubes.

In view of the 36:1 gear ratio between the output shaft and synchro follow-up link the error of 10 min. produces an error voltage of 6 volts r.m.s. This voltage is applied to the input terminals of the error differentiating network (notch filter). It was specified that this should be a parallel-T network. The details of this filter are not worked out here, since the subject was fully covered in Chap. VII. The relative frequency bandwidth being equal to $x_b = 0.3$, as computed above in Sec. 5, the output voltage of the notch filter network is equal to

$$e_c = \frac{x_b}{4} = \frac{0.3}{4} = 0.075 \text{ volt output per volt input.}$$

For an error of 10 min., or an error signal voltage of 6 volts, this corresponds to a voltage of $6 \times 0.075 = 0.45$ volt appearing at the output terminals of the differentiating network.

A low-pass filter being inserted between this network and the amplifier,[1] producing a two-to-one gain reduction, the amplifier gain is calculated.

$$\text{Amplifier gain} = \frac{105 \times 2}{0.45} = 466.$$

This can be readily obtained by means of the two-stage amplifier shown in the figure, using type 6SN7 twin triodes and a suitable rectified d-c power supply. The design of this amplifier is conventional, and is not developed here, since it offers no particular difficulty.

Regulators and Stabilizers.—As mentioned earlier, position control is only one of many possible uses of control mechanisms. The same principles that govern the operation of position-control servomechanisms also form the basis of other applications of control devices. Suitable interpretation of the equations discussed in earlier parts of this book will make these adaptable to other problems and conditions.

To illustrate this point, reference may be made to Fig. 10.5, which shows a servomechanism of the same pattern as studied before. It comprises an input member, an output member, a differential device, and a controller. When the system is at rest, with the input member in the position A, the output member assumes the corresponding position A'. No error signal then actuates the controller, and the latter is inoperative.

For the usual type of position control, as discussed previously, the input member is displaced to the position B, and the resulting difference between the positions B and A' of the input and output members is

[1] The purpose of this low-pass filter is to suppress harmonics from the signal voltage, and introduce a phase shift to compensate for the phase shifts produced by the transformers included in the amplifier and power control circuits.

translated into an error signal by the differential device. This error signal actuates the controller, which then drives the output member into the position B' corresponding to the new input position B. This corrects the input-output position error to zero, eliminates the differential error signal, and stops the operation of the controller. Thus, the output member follows the displacement of the input member. The speed and accuracy with which the output member is driven into position correspondence with the input member and the possibility for the operation

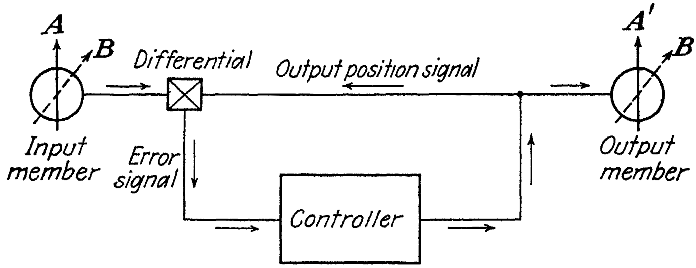

FIG. 10.5.—Operation of position-control servomechanism.

to take place with or without oscillations of the output member depend on the constants (friction, inertia, gain, etc.) of the system. They depend also on the nature of the function,[1] which relates the output force or torque of the controller to the error signal produced by the differential device.

Now suppose that the input member instead of being moved is kept stationary in its original position A. Under normal conditions the output member is then in the corresponding position A'. Let the output member be subjected to a temporary external disturbing force. For example, the load connected to the output shaft may be a radio antenna exposed to variable wind currents; or it may be a mass related to the hull of a ship or aircraft rolling or pitching under the effect of water or air currents and waves. If, under these conditions, the output member is displaced from the position A' to the position B', Fig. 10.5, an error signal is generated by the differential device, proportional to the difference between the new output position B' and the original output position A', as determined by the fixed input position A. The controller, accordingly, develops a force or torque so directed as to reduce the error to zero by driving the output member back to the correspondence position A'. This is accomplished with or without oscillations of the output member and with a speed and accuracy depending on the same factors as before.

[1] Three types of functions were studied in the preceding chapters, according to which the controller torque is proportional to either the error signal, or its first derivative, or its time integral. Other functions, sometimes more complicated, may be used, depending on the purpose to be achieved.

In this latter mode of operation the control device performs the function of a *regulator* or *stabilizer*, and maintains the position or condition of the output member at the fixed standard established by the setting of the input member. Obviously, the same equations as were derived previously will apply in this type of application as well.

Similar principles may also be used for the control of quantities other than the relative positions of the input and output members. Among the quantities that may be controlled by such methods, mention may be made of temperature, pressure, level, speed, acceleration of a fluid or process,[1] and so on. On the other hand, mechanical displacement is not necessarily associated with the operation of the mechanism, for instance when the controlled quantity that characterizes the condition of the output member is an electric current or voltage. For, as explained, position control by means of the devices described in the preceding chapters is obtained by translating the position error into an electrical signal and by translating the controller output voltage back into a mechanical force capable of altering the position of the system. This double translation process can be eliminated when purely electrical quantities are to be dealt with.

Equation Correspondence.—The following paragraphs illustrate the manner in which the equations established previously for position-control servos may be used in conjunction with other servomechanisms. However, before giving some concrete examples, these equations will be written in dimensionless form to make them more readily adaptable to a variety of servo systems.

The method will consist in first setting up the differential equation of the particular servomechanisms considered. Secondly, in this equation the coefficients of the variable and of its successive derivatives are compared with and identified with, respectively, the corresponding coefficients in a related equation given below which must be of the same order[2] and form as the equation of the servomechanism considered. The coefficients, having then been expressed in terms of those of the related equation, the performance of the servo is in every respect similar to that of the system corresponding to the related equation, and will have the same transient and steady-state errors, degree of stability, etc.

While the performance of servomechanisms was studied previously in terms of the input-output error expressed as a function of time, it is

[1] Control of light flux is mentioned by H. L. Hazen, Design and Test of a High-performance Servomechanism, *J. Franklin Inst.*, vol. 218, p. 543, November, 1934.

[2] Certain types of servos may lead to differential equations of higher order than were encountered in this book. As will be shown later, this does not prevent an evaluation of their effective damping constant and degree of stability being made, through application of the methods of transfer function analysis.

often convenient to work from equations which correlate the input and output variables. These equations are established below for the same types of servos that were discussed before. For convenience, all parameters used are dimensionless, and the coefficient of the highest derivative term is made equal to unity.

Equation for Inertialess Servomechanism.—In many cases the differential equation of a servomechanism corresponds to that of a servo in which the inertia term is negligibly small. The equation of an inertialess system is therefore derived below.

Consider a servo system comprising output friction only. The retarding and driving torques are then equal at all times. Using the same symbols as before, this may be written

$$\begin{cases} F\dfrac{d\theta_o}{dt} = K\theta. & (10.1) \\ \theta = \theta_i - \theta_o. & (10.2) \end{cases}$$

Substituting in Eq. (10.1) the value of θ_o derived from Eq. (10.2), there comes

$$\begin{cases} \theta_o = \theta_i - \theta & (10.3) \\ F\dfrac{d\theta_i}{dt} - F\dfrac{d\theta}{dt} = K\theta, & (10.4) \end{cases}$$

or

$$F\frac{d\theta}{dt} + K\theta = F\frac{d\theta_i}{dt}. \tag{10.5}$$

Dividing through by F, and setting for simpler writing

$$v = \frac{K}{F} \tag{10.6}$$

Eq. (10.5) becomes

$$\frac{d\theta}{dt} + v\theta = \frac{d\theta_i}{dt}. \tag{10.7}$$

For a step velocity input function, the transient term of the solution of this equation is obtained by setting the right-hand member equal to zero.

$$\frac{d\theta}{dt} + v\theta = 0. \tag{10.8}$$

Assuming a solution of the form

$$\theta = A\epsilon^{pt}, \tag{10.9}$$

Eq. (10.8) becomes

$$Ap\epsilon^{pt} + Av\epsilon^{pt} = 0; \tag{10.10}$$

or, dividing through by $A\epsilon^{pt}$,

$$p = -v. \tag{10.11}$$

The transient solution, Eq. (10.9), is then expressed

$$\theta = A\epsilon^{-vt}. \tag{10.12}$$

In the steady-state operating condition, the error is constant, and Eq. (10.7) then becomes

$$v\theta = \frac{d\theta_i}{dt}. \tag{10.13}$$

The right-hand member being equal to the input speed ω_1, Eq. (10.13) becomes

$$v\theta = \omega_1, \tag{10.14}$$

and the steady-state solution is

$$\theta = \frac{\omega_1}{v}. \tag{10.15}$$

The complete solution of Eq. (10.7) is then the sum of the transient and steady-state solutions, Eqs. (10.12) and (10.15).

$$\theta = \frac{\omega_1}{v} + A\epsilon^{-vt}. \tag{10.16}$$

Since the initial condition of the system is that the error θ is equal to zero

$$\theta = 0, \qquad t = 0 \tag{10.17}$$

it follows from Eq. (10.16) that

$$A = -\frac{\omega_1}{v}, \tag{10.18}$$

and the total solution is expressed

$$\theta = \frac{\omega_1}{v}(1 - \epsilon^{-vt}). \tag{10.19}$$

This can be written in dimensionless form

$$\theta\frac{v}{\omega_1} = 1 - \epsilon^{-vt}. \tag{10.20}$$

Equation for Viscous-damped Servomechanism.—A combination of Eqs. (4.1) and (4.2) of Chap. IV gives

$$K\theta_i = J\frac{d^2\theta_o}{dt^2} + F\frac{d\theta_o}{dt} + K\theta_o, \tag{10.21}$$

or, dividing through by J in order to make the coefficient of the highest derivative term equal to unity,

$$\frac{K}{J}\,\theta_i = \frac{d^2\theta_o}{dt^2} + \frac{F}{J}\frac{d\theta_o}{dt} + \frac{K}{J}\,\theta_o. \tag{10.22}$$

From Eqs. (4.50), (4.58), and (4.61)

$$\frac{K}{J} = \omega_n{}^2 \tag{10.23}$$

$$\frac{F}{J} = \frac{2c\sqrt{KJ}}{J} = 2c\omega_n. \tag{10.24}$$

Substituting these expressions in Eq. (10.22), this becomes

$$\omega_n{}^2\theta_i = \frac{d^2\theta_o}{dt^2} + 2c\omega_n\frac{d\theta_o}{dt} + \omega_n{}^2\theta_o. \tag{10.25}$$

Equation for Error-rate-damped Servomechanism.—A combination of Eqs. (5.2) and (5.3) of Chap. V gives

$$L\frac{d\theta_i}{dt} + K\theta_i = J\frac{d^2\theta_o}{dt^2} + L\frac{d\theta_o}{dt} + K\theta_o, \tag{10.26}$$

or, dividing through by J in order to make the coefficient of the highest derivative term equal to unity,

$$\frac{L}{J}\frac{d\theta_i}{dt} + \frac{K}{J}\,\theta_i = \frac{d^2\theta_o}{dt^2} + \frac{L}{J}\frac{d\theta_o}{dt} + \frac{K}{J}\,\theta_o. \tag{10.27}$$

Substituting for the coefficients their expressions found in Chap. V,

$$\frac{K}{J} = \omega_n{}^2 \tag{10.28}$$

$$\frac{L}{J} = \frac{2c\sqrt{KJ}}{J} = 2c\omega_n, \tag{10.29}$$

Eq. (10.27) becomes

$$2c\omega_n\frac{d\theta_i}{dt} + \omega_n{}^2\theta_i = \frac{d^2\theta_o}{dt^2} + 2c\omega_n\frac{d\theta_o}{dt} + \omega_n{}^2\theta_o. \tag{10.30}$$

Equation for Combined Viscous-damped and Error-rate-damped Servomechanism.—A combination of Eqs. (6.1) and (6.2) of Chap. VI gives

$$L\frac{d\theta_i}{dt} + K\theta_i = J\frac{d^2\theta_o}{dt^2} + (F + L)\frac{d\theta_o}{dt} + K\theta_o; \tag{10.31}$$

or, dividing through by J in order to make the coefficient of the highest

derivative term equal to unity,

$$\frac{L}{J}\frac{d\theta_i}{dt} + \frac{K}{J}\,\theta_i = \frac{d^2\theta_o}{dt^2} + \frac{F+L}{J}\frac{d\theta_o}{dt} + \frac{K}{J}\,\theta_o. \qquad (10.32)$$

Substituting for the coefficients their expressions as found before,

$$\left\{\begin{aligned} \frac{K}{J} &= \omega_n^2 & (10.33)\\ \frac{L}{J} &= 2c(1-r)\omega_n & (10.34)\\ \frac{F+L}{J} &= 2c\omega_n, & (10.35) \end{aligned}\right.$$

Eq. (10.32) is written

$$2c(1-r)\omega_n\frac{d\theta_i}{dt} + \omega_n^2\theta_i = \frac{d^2\theta_o}{dt^2} + 2c\omega_n\frac{d\theta_o}{dt} + \omega_n^2\theta_o. \qquad (10.36)$$

Equation for Integral Control Servomechanism.—Combining Eqs. (8.1) and (8.2) of Chap. VIII, and differentiating, there comes

$$L\frac{d^2\theta_i}{dt^2} + K\frac{d\theta_i}{dt} + N\theta_i = J\frac{d^3\theta_o}{dt^3} + (F+L)\frac{d^2\theta_o}{dt^2} + K\frac{d\theta_o}{dt} + N\theta_o. \quad (10.37)$$

Dividing through by J, and substituting for the coefficients their expressions, as found in Chap. VIII, this equation becomes

$$\begin{aligned} 2c(1-r)\omega_n\frac{d^2\theta_i}{dt^2} &+ \omega_n^2\frac{d\theta_i}{dt} + s\omega_n^3\theta_i \\ &= \frac{d^3\theta_o}{dt^3} + 2c\omega_n\frac{d^2\theta_o}{dt^2} + \omega_n^2\frac{d\theta_o}{dt} + s\omega_n^3\theta_o. \quad (10.38) \end{aligned}$$

Extension of Servo-positioning Equations to Other Control Devices. Examples are given below to illustrate the use of the preceding equations in conjunction with control devices other than those intended for position control.

Example 1. *Plastic Molding Press.*—In a plastic molding press it is desirable to be able to subject the plastic to a specific cycle of temperatures accurately maintained during some given curing period and then suddenly changed to some other value. For this purpose the mold containing the plastic is arranged so that either cold water or superheated steam may be circulated in the die. A thermocouple immersed in the plastic mass indicates the temperature of the latter. The thermocouple connects to a controller, which is set to the temperature at which the plastic is to be maintained. The controller operates two valves, through which either the water or the steam is admitted to the die, depending on whether the temperature of the plastic exceeds or falls below the assigned

temperature. The problem is to determine the operating characteristics of the system.

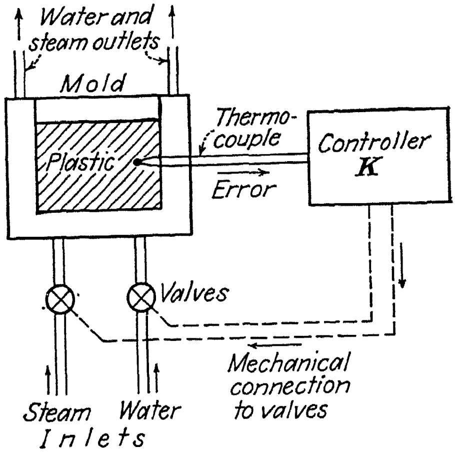

FIG. 10.6.—Plastic molding press.

In order to write the differential equation of the system, the following notation will be used:

T_o = actual instantaneous temperature of the plastic.
T = temperature of the die.
T_i = temperature to be maintained in the plastic.
p = rate at which energy is taken away from or supplied to the die.
t = time.
A, B, and C = proportionality constants characterizing the system, as described below.

The conditions of operation of the system are as follows:

1. The rate of temperature variation of the plastic is proportional to the temperature difference between the plastic and the die walls (Newton's law).

$$\frac{dT_o}{dt} = A(T - T_o). \tag{10.39}$$

2. The rate of temperature variation of the die is proportional to the heat transferred to or from the die.

$$\frac{dT}{dt} = Bp. \tag{10.40}$$

3. The controller is so arranged that the power to or from the die is proportional to the difference between the temperature which it is desired to maintain in the plastic and the actual temperature of the latter.

$$p = C(T_i - T_o). \tag{10.41}$$

Differentiating Eq. (1C.39) gives

$$\frac{d^2T_o}{dt^2} = A\frac{dT}{dt} - A\frac{dT_o}{dt}. \tag{10.42}$$

Substituting in this equation the expression for dT/dt, as given in Eq. (10.40), there comes

$$\frac{d^2T_o}{dt^2} = ABp - A\frac{dT_o}{dt}. \tag{10.43}$$

Substituting for p its expression, as shown in Eq. (10.41), Eq. (10.43) becomes

$$\frac{d^2T_o}{dt^2} = ABC(T_i - T_o) - A\frac{dT_o}{dt}, \tag{10.44}$$

which can be written

$$\frac{d^2T_o}{dt^2} + A\frac{dT_o}{dt} + ABCT_o = ABCT_i. \tag{10.45}$$

By inspection, this equation appears to be of the same form and order as Eq. (10.25) for a viscous-damped servomechanism, in which

$$\begin{cases} A = 2c\omega_n & (10.46) \\ ABC = \omega_n^2, & (10.47) \end{cases}$$

or

$$\begin{cases} \omega_n = \sqrt{ABC} & (10.48) \\ c = \dfrac{A}{2\omega_n} = \dfrac{1}{2}\sqrt{\dfrac{A}{BC}}. & (10.49) \end{cases}$$

Certain of the constants which appear here are determined by the particular application of the system. Others are chosen at the discretion of the designer, who can thus obtain any desired value of effective damping ratio c and speed of response ω_n to sudden temperature changes. In accordance with these values the performance of the system (rate of decay of the transient error, magnitude of the steady-state error, etc.) will be represented by the curves of Fig. 4.6.

Example 2. *Frequency-regulating System.*—Consider a conventional triode oscillator, Fig. 10.7, such as used in many types of radio apparatus. It consists of a three-electrode vacuum tube, or triode, T, the grid and plate circuits of which are properly connected to a *tank circuit* composed of an inductance coil A and capacitor B. The frequency of the alternating voltage generated by this arrangement is primarily determined by the inductance and capacitance of the tank circuit, and to some extent by the operating (d-c) voltages of the triode. The frequency may drift or vary from its intended fixed value, following voltage variations of

the tube or temperature changes that cause various components of the system to expand or contract slightly, thereby changing their electrical dimensions. The object of the control device described and discussed below is to correct any such frequency changes as may occur and to maintain the frequency at a fixed, predetermined value.

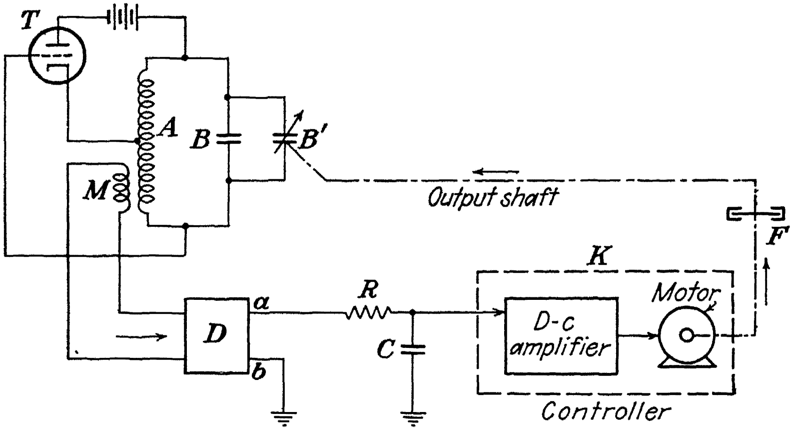

FIG. 10 7 —Frequency-regulating system.

For this purpose, a sample of the alternating voltage generated in the tank circuit AB is picked up inductively by a coil M, which is coupled to the tank circuit coil A. The voltage induced in the coil M is then applied to a *discriminator* D, which may be concisely described as a tuned rectifier, producing a unidirectional voltage at its output terminals a and b. The tuning of the discriminator is set to the frequency value that it is desired to maintain in the oscillator tank circuit AB. When the operating frequency of the triode T (and therefore, also, the frequency of the voltage applied to the discriminator through the pick-up coil M) is equal to the desired frequency, the output voltage of the discriminator (terminals a and b) is zero. When the triode operating frequency departs from the desired value, a unidirectional (d-c) voltage appears at the discriminator output terminals a and b. This voltage is positive or negative, depending, respectively, on whether the triode operating frequency is greater or smaller than the desired frequency. Moreover, this discriminator output voltage has a magnitude that is directly proportional to the difference between the actual and the desired operating frequencies of the triode oscillator.

The discriminator output voltage is thus an *error voltage*, which indicates the magnitude and direction of the departure of the triode operating frequency from the assigned value. This is illustrated in Fig. 10.8.

The positive or negative (d-c) output voltage of the discriminator is

then amplified, and controls the operation (speed and direction of rotation) of a direct-current motor. This, in turn, drives a variable *trimmer* capacitor B' connected in parallel with the main, fixed capacitor B of the oscillator tank circuit. By turning the movable plates of this trimmer capacitor by a small angle one way or another, the over-all capacitance of the tank circuit is altered, and the frequency of the alternating voltage generated by the triode T is changed accordingly. For a proper connection polarity of the discriminator to the controller, the motor will operate the trimmer capacitor in such a direction as to correct the triode operating frequency to its fixed assigned value, thereby returning the discriminator output voltage to zero.

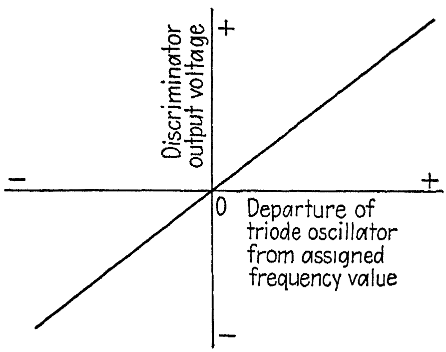

FIG. 10.8.—Discriminator voltage-frequency characteristic.

It should be noted that the input voltage to the discriminator, which is the voltage induced in the pick-up coil M, is a high-frequency (oscillator frequency) alternating voltage. The rectified output voltage developed at the terminals a and b is a succession of half-waves or half-sinusoids (positive or negative, depending on whether the triode operating frequency is greater or smaller than the assigned value). In order to smooth out this voltage into a truly continuous voltage (positive or negative, as the case may be), suitable for operation in the d-c amplifier and motor of the controller, a filter circuit is inserted in the error voltage channel, between the discriminator and controller.

This filter circuit is composed of a series resistor **R** followed by a shunt capacitor **C**. While such a filter will stop, or greatly attenuate, voltages of higher frequencies (frequency of the oscillations generated by the triode oscillator, and harmonics produced by the rectifier in the discriminator output voltage), it will pass continuous voltage and such low-frequency voltage fluctuations as follow the comparatively slow variations of the triode oscillator frequency. The transfer characteristic of the filter is represented in Fig. 10.9, which shows, against frequency the ratio of the filter output voltage to the filter input voltage.

At a frequency ω_x equal to

$$\omega_x = \frac{1}{RC} \tag{10.50}$$

the filter output voltage is equal to 0.707 times the filter input voltage. The triode oscillator frequency, which is the controlled frequency, is generally substantially greater than ω_x. However, if the controlled frequency is low, it may be necessary, in order to obtain a smooth filter output voltage, to make the cutoff frequency ω_x so low as to influence

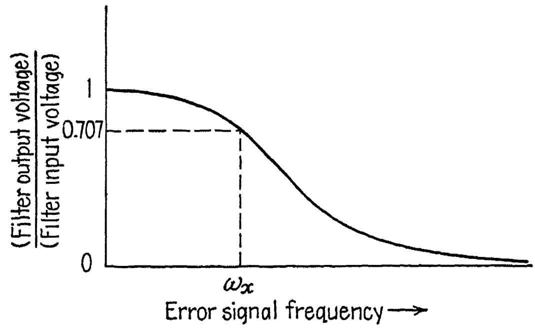

FIG. 10.9.—Transfer characteristic of low-pass filter network.

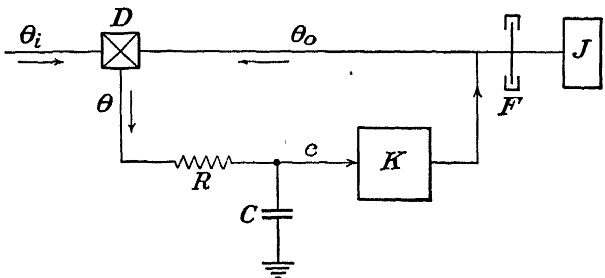

FIG. 10.10.—Position-control servomechanism equivalent of frequency-regulating system shown in Fig. 10.7.

the stability and speed of response of the system. It is this problem that will be considered below.

The closed-cycle frequency control system of Fig. 10.7 may be likened, functionally, to a simple viscous output damped position-control servomechanism, in which a resistance-capacitance filter network is inserted in the error signal channel. Such an equivalent position-control servo is represented schematically in Fig. 10.10 in the same manner and with the same symbols as before. The natural frequency ω_n and damping ratio c of this servo, with the resistance-capacitance filter network removed,

was expressed in Chap. IV.

$$\begin{cases} \omega_n = \sqrt{\dfrac{\mathrm{K}}{\mathrm{J}}} & (10.51) \\ \mathrm{c} = \dfrac{\mathrm{F}}{2\sqrt{\mathrm{KJ}}}. & (10.52) \end{cases}$$

The first step in analyzing the system is to write the differential equation of its motion. Because of the presence of the filter network **RC**, which distinguishes the system from that previously studied in Chap. IV, it is necessary, at first, to obtain two separate equations, one relating the output motion to the output voltage of the network (or input voltage to the controller), and the other relating the output voltage of the network to the error θ. The network output voltage will be designated by e, and may be considered to have the same dimensions as θ. Then, e can be eliminated between the two equations, and the differential equation may be obtained by relating the error θ and the output motion θ_o.

The equation between e and θ_o will be identical with that obtained between θ and θ_o for a simple viscous output damped servomechanism. Thus, from Eq. (4.1) of Chap. IV,

$$\mathrm{K}e = \mathrm{J}\frac{d^2\theta_o}{dt^2} + \mathrm{F}\frac{d\theta_o}{dt}. \tag{10.53}$$

The equation relating θ and e may be written

$$\theta = e + \mathrm{RC}\frac{de}{dt}. \tag{10.54}$$

The expression for e, as given by Eq. (10.53), and that for de/dt, as obtained by differentiating Eq. (10.53) being substituted in Eq. (10.54), the latter becomes

$$\mathrm{K}\theta = \mathrm{RCJ}\frac{d^3\theta_o}{dt^3} + (\mathrm{J} + \mathrm{RCF})\frac{d^2\theta_o}{dt^2} + \mathrm{F}\frac{d\theta_o}{dt}. \tag{10.55}$$

Since, from the definition of the error,

$$\theta_o = \theta_i - \theta, \tag{10.56}$$

the equation between θ and θ_i is obtained by substituting Eq. (10.56) in Eq. (10.55).

$$\mathrm{RCJ}\frac{d^3\theta}{dt^3} + (\mathrm{J} + \mathrm{RCF})\frac{d^2\theta}{dt^2} + \mathrm{F}\frac{d\theta}{dt} + \mathrm{K}\theta$$
$$= \mathrm{RCJ}\frac{d^3\theta_i}{dt^3} + (\mathrm{J} + \mathrm{RCF})\frac{d^2\theta_i}{dt^2} + \mathrm{F}\frac{d\theta_i}{dt}. \tag{10.57}$$

This equation could be solved by methods similar to those previously applied to the integral control problem (Chap. VIII), since this was also found to lead to a third-order differential equation. However, because both problems are governed by third-order differential equations, many of the results obtained from the discussion of the integral control system may be used directly here for determining the stability and speed of response of the present system. At the same time, it should be noted that although both systems lead to third-order equations they may differ greatly in type of performance.[1] Thus, specific characteristics may be radically different.

Like the other viscous output damped servomechanisms studied in Chap. IV, the viscous output damped servo shown in Fig. 10.10 will have a transient error and a steady-state error when its input member is subjected to a step velocity function. A similar condition would obtain in the system of Fig. 10.7 if the oscillator frequency were suddenly tending to vary continuously at a constant rate in a fixed direction.[2] A variation of this type, however, is generally not encountered in practice, and only the transient error is of practical interest, particularly in that it will denote the stability and speed of response of the system.

The transient solution of Eq. (10.57) may be obtained by setting the right-hand member of this equation equal to zero.

$$\mathbf{RCJ}\,\frac{d^3\theta}{dt^3} + (\mathbf{J} + \mathbf{RCF})\,\frac{d\theta^2}{dt^2} + \mathbf{F}\,\frac{d\theta}{dt} + \mathbf{K}\theta = 0. \tag{10.58}$$

A more convenient form is obtained by dividing through by **RCJ**, and writing the coefficients in terms of the parameters given in Eqs. (10.50), (10.51), and (10.52). Equation (10.58) then becomes

[1] The differential equation (10.57) may be compared with the complete equation for the integral control system, as obtained by differentiating Eq. (8.3) of Chap. VIII.

$$J\,\frac{d^3\theta}{dt^3} + (L + F)\,\frac{d^2\theta}{dt^2} + K\,\frac{d\theta}{dt} + N\theta = J\,\frac{d^3\theta_i}{dt^3} + F\,\frac{d^2\theta_i}{dt^2}. \tag{A}$$

It is obvious that no values of the coefficients of Eq. (A) can make this equation the same as Eq. (10.57), since the latter contains a term in $d\theta_i/dt$ that does not appear in equation (A).

[2] Suppose that the capacitance B of the oscillator tank circuit, Fig. 10.7, is suddenly started decreasing, so as to raise the oscillator frequency at a constant rate. The servo system will then operate in such a manner as steadily to increase the capacitance of the trimmer capacitor B', driven by the controller, in order to maintain the oscillator frequency at its fixed value, as assigned by the discriminator setting. However, after the transient error of the starting period will have died, a steady-state error will remain, causing the oscillator frequency to assume a value slightly higher than the assigned value, by an amount proportional to the oscillator tuning variation speed.

$$\frac{d^3\theta}{dt^3} + (\boldsymbol{\omega}_x + 2\mathbf{c}\boldsymbol{\omega}_n)\frac{d^2\theta}{dt^2} + 2\mathbf{c}\boldsymbol{\omega}_x\boldsymbol{\omega}_n\frac{d\theta}{dt} + \boldsymbol{\omega}_x\boldsymbol{\omega}_n{}^2\theta = 0. \tag{10.59}$$

Comparing this to the analogous equation (8.17) for the integral control case,

$$\frac{d^3\theta}{dt^3} + 2c\omega_n\frac{d^2\theta}{dt^2} + \omega_n{}^2\frac{d\theta}{dt} + s\omega_n{}^3\theta = 0, \tag{8.17}$$

it may be seen that although all the corresponding coefficients are different the same number of terms and of parameters are present in both equations. Therefore, for any set of values of $\mathbf{c}$, $\boldsymbol{\omega}_n$, and $\boldsymbol{\omega}_x$, a corresponding set of values of c, ω_n, and s may be found by equating the coefficients of θ and each of its derivatives as follows:

$$\begin{cases} \boldsymbol{\omega}_x + 2\mathbf{c}\boldsymbol{\omega}_n = 2c\omega_n & (10.60)\\ 2\mathbf{c}\boldsymbol{\omega}_x\boldsymbol{\omega}_n = \omega_n{}^2 & (10.61)\\ \boldsymbol{\omega}_x\boldsymbol{\omega}_n{}^2 = s\omega_n{}^3. & (10.62)\end{cases}$$

Defining, for simpler writing, the ratio

$$\mathbf{x} \equiv \frac{\boldsymbol{\omega}_n}{\boldsymbol{\omega}_x} \tag{10.63}$$

the above relations, Eqs. (10.60), (10.61), and (10.62), may be solved for $\mathbf{c}$, $\boldsymbol{\omega}_n$, and $\mathbf{x}$,

$$\begin{cases} \mathbf{c} = \sqrt{\dfrac{c \pm \sqrt{c^2 - 1}}{4s}} & (10.64)\\ \mathbf{x} = (c \pm \sqrt{c^2 - 1})^{3/2}\sqrt{s} & (10.65)\\ \boldsymbol{\omega}_n = \sqrt{s(c \pm \sqrt{c^2 - 1})}. & (10.66)\end{cases}$$

These same relations, Eqs. (10.60), (10.61), and (10.62), may be solved for c, ω_n, and s

$$\begin{cases} c = \dfrac{1}{2}\left(\dfrac{1}{\sqrt{2\mathbf{cx}}} + \sqrt{2\mathbf{cx}}\right) & (10.67)\\ \omega_n = \boldsymbol{\omega}_n\sqrt{\dfrac{2\mathbf{c}}{\mathbf{x}}} & (10.68)\\ s = \sqrt{\dfrac{\mathbf{x}}{8\mathbf{c}^3}} & (10.69)\end{cases}$$

From the values of c, ω_n, and s thus obtained, the equations and graphs derived in Chap. VIII[1] may be used to compute g, ω_g, and h for the complete system. It will be recalled that ω_g is the natural frequency of the oscillating part of the transient, while g is the ratio of the effective

[1] Equations (8.46), (8.47), and (8.48), and Figs. 8.4 and 8.5.

damping to that required for critical damping, and h is the coefficient, which denotes the rate of decay of the simple exponential term of the general transient. The form of the general transient solution is as given in Chap. VIII.

$$\theta = A\epsilon^{-h\omega_g t} + \epsilon^{-g\omega_g t}(B_1 \cos \omega_g \sqrt{1 - g^2}\, t + B_2 \sin \omega_g \sqrt{1 - g^2}\, t). \quad (8.42)$$

If it were desired to obtain the exact solution for a given input function, it would be necessary to find the steady-state solution and evaluate the constants. However, this procedure may be dispensed with if it is desired to ascertain only the general behavior of the system, since the damping ratios and natural frequencies of the transients have been determined.

Thus, although the exact problem considered here has not previously been treated, it is possible to discuss and study the performance of the system, by using the equations and curves established previously for a servomechanism that, although entirely different and employing different means of control, was governed by a differential equation of the same order as the system considered.

Transfer Function Analysis.—Such a comparison of the differential equation of a new servomechanism with one of the equations previously established would not have been possible if the new system had been more complicated and led to a higher order differential equation than studied before. It will therefore now be shown how a transfer function analysis may help to determine the operating stability of the system; the method is applicable irrespective of the degree of complexity of the system.

The procedure consists, after having established the differential equation of the system relating, for instance, the error θ and output motion θ_o (or its equivalent), in assuming a sinusoidal error function and plotting the locus of the output vector, taking the error vector as a reference. This was done in detail in Chap. IX for the simple type servos considered before, and will be carried through, as a further illustration, for the system of Fig. 10.7 (or Fig. 10.10) here discussed.

Using the same notation as in Chap. IX, the sinusoidal error function of unit amplitude and the resulting output function are written

$$\begin{cases} \theta = \epsilon^{j\omega t} & (10.70) \\ \theta_o = B\epsilon^{j\mu}\epsilon^{j\omega t}. & (10.71) \end{cases}$$

Substituting these expressions in the differential equation (10.55) of the system, the output-error transfer function is obtained.

$$\frac{\theta_o}{\theta} = B\epsilon^{j\mu} = \frac{1}{-(1 + 2\mathbf{c}\mathbf{x})d^2 - j(\mathbf{x}d^3 - 2\mathbf{c}d)}. \quad (10.72)$$

As in Chap. IX, this expression allows the locus of the output vector to be plotted, for varying values of the relative frequency d, after particular values have been assigned to the parameters of the system. Let then

$$\mathbf{c} = 0.582$$
$$\mathbf{x} = 0.859$$

which are chosen so that the effective damping ratio g of the transient would be equal to 0.25. This is done in order to compare the behavior of the system with that of other systems previously studied, which had this same damping ratio 0.25.

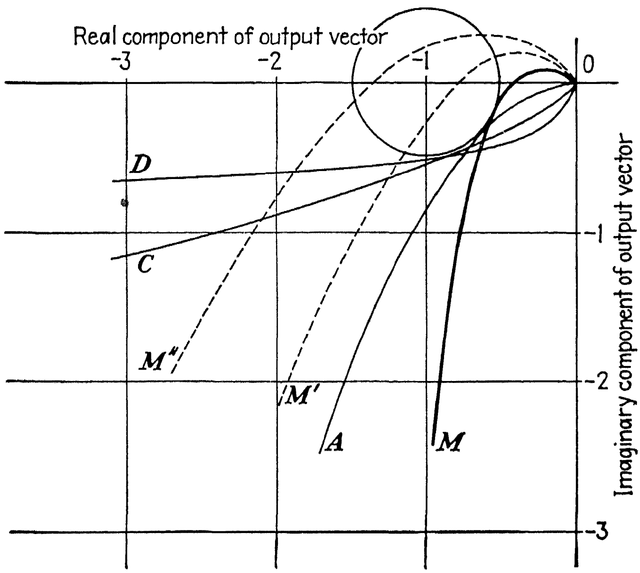

FIG. 10.11.—Comparison of transfer functions of different servomechanisms having a same damping ratio.

The actual plot of Eq. (10.72) is shown in Fig. 10.11 by the heavy line curve M. It was stated in Chap. IX that a good estimate of the stability of the system may be obtained by determining the radius of the circle tangent to the $Be^{j\mu}$ curve and having as its center the -1 point of the horizontal axis. This circle is shown in Fig. 10.11. For the purpose of comparison, the diagram also shows curves A, C, and D, which are reproductions of the output-error transfer function curves previously found for the following systems.

Curve A: viscous output damped servo with damping ratio $c = 0.25$.

Curve D: error-rate-damped servo with $c = 0.25$ (and $r = 0$).

Curve C: integral control servo with $c = 0.4$, $r = 0.625$, and $s = 0.29$, giving an effective damping ratio $g = 0.25$.

The systems represented by the four curves A, D, C, and M of Fig. 10.11 thus have all the same effective damping ratio 0.25. Mere inspection of the diagram shows that these curves are all tangent to practically the same circle centered on the -1 point. This substantiates the criterion established in the preceding chapter, and it indicates that all four systems have approximately the same degree of stability. The transient characteristics (oscillation amplitude, duration, rate of decay, and speed of response) of the new system are then also comparable to those of the three other systems previously studied, as represented graphically by the error-time curves, which were given in the preceding chapters of this book.

It should be noted that contrary to the other curves of Fig. 10.11 the curve M of the system considered here passes, at first, in the second quadrant of the diagram, then crosses the horizontal axis between the origin and the -1 point, and finally extends down into the third quadrant. The possibility of obtaining curves of this type was mentioned in the preceding chapter, where a rule was also given for determining whether or not the system would then be subject to continuous oscillation, or hunting. Since, in the present instance, the curve M passes *below* the -1 point of the horizontal axis, the system will be stable.

However, if the gain of the amplifier shown in Fig. 10.7 were increased, the magnitude of the output displacement corresponding to a unit error signal would increase accordingly. The $B\epsilon^{j\mu}$ curve of the system may then assume a shape such as shown by the dotted line curve M' or M''. In the latter case (curve M'') the system would be unstable and would hunt continuously.

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